1. (a) Write \( x = (x_1, x_2), \ y = (y_1, y_2), \ z = (z_1, z_2) \)

- Since \( |x_1 - y_1|, |x_2 - y_2| \geq 0 \) we have that \( d_m(x, y) \geq 0 \)
- If \( x = y \), then \( d_m(x, y) = \max \{0, 0\} = 0 \).

On the other hand, if \( \max \{ |x_1 - y_1|, 1, |x_2 - y_2| \} = 0 \), then \( x_1 = y_1 \) and \( x_2 = y_2 \). It follows that \( x = y \) when \( d_m(x, y) = 0 \).

- \( d_m \) is clearly symmetric.

- For the triangle inequality, \( d_m(x, y) = \max \{ 1, x_1 - y_1, 1, x_2 - y_2 \} \)
  \[ = \max \{ 1, 1, |x_1 - y_1|, |x_2 - y_2| \} \]
  \[ \leq \max \{ 1, 1, |x_1 - y_1|, |x_2 - y_2| \} + \max \{ 1, 1, |x_2 - y_1|, |x_2 - y_2| \} \]
  \[ = \max \{ 1, 1, |x_1 - y_1|, |x_2 - y_2| \} \]
  \[ = d_m(x, z) + d_m(z, y) \]

(b)

\[
\begin{align*}
\text{(d)} & \quad \text{d}_E \\
& \quad \text{(c)} \quad \text{d}_T \\
& \quad \text{(d)} \quad \text{d}_M \\
\end{align*}
\]

2. (a) Write \( f(x) = ax^2 + bx + c \geq 0 \). Since \( f(x) \geq 0 \)

for all \( x \in \mathbb{R} \), \( a > 0 \) and \( f \) attains its minimum at \( \mu = -\frac{b}{2a} \). It follows that \( f(\mu) = c - \frac{b^2}{4a} \geq 0 \)

and so \( \frac{b^2 - c}{4a} \leq 0 \). Since \( a > 0 \) we conclude

\[ \frac{b^2 - 4ac}{4a} \leq 0 \]
(b) Observe that \((\sum y_i^2) \lambda^2 - (2\sum x_i y_i) \lambda + (\sum x_i^2) = \sum (x_i - \lambda y_i)^2\) is a non-negative quadratic polynomial. By part (a),

\[4(\sum x_i y_i)^2 - 4\sum y_i^2 \sum x_i^2 \leq 0.\]

It follows that \((\sum x_i y_i)^2 \leq \sum x_i^2 \sum y_i^2\).

The result now follows by taking square roots.

(c) \[\left[ d(x, z) + d(z, y) \right]^2 = d^2(x, z) + 2d(x, z)d(z, y) + d^2(z, y)\]

\[= \sum (x_i - z_i)^2 + 2\sqrt{\sum (x_i - z_i)^2} \sqrt{\sum (z_i - y_i)^2} + \sum (z_i - y_i)^2\] (Cauchy-Schwarz)

\[\geq \sum (x_i - z_i)^2 + 2\sum (x_i - z_i)(z_i - y_i) + \sum (z_i - y_i)^2\]

\[= \sum (x_i - 2x_i y_i + y_i^2) + 2\sum (x_i z_i - x_i y_i - y_i z_i + z_i^2) + \sum (z_i - 2y_i z_i + y_i^2)\]

\[= \sum (x_i^2 - 2x_i y_i + y_i^2) = \sum (x_i - y_i)^2 = d^2(x, y).\]

Now take square roots.
3. \( \text{let } U = \{ U : \forall U \subseteq X \text{ and } U \text{ is open in } X \} \).

Pick \( x \in U \). Then \( x \in U \) for some \( U \in U \). Since \( U \) is open in \( X \), there exists \( r > 0 \) such that \( B(x, r) \subseteq U \).

Now we have \( B(x, r) \subseteq U \subseteq Y \) and \( x \in \text{int } Y \).

\[ \therefore U \subseteq \text{int } Y. \]

If \( x \in \text{int } Y \), then there exists \( r > 0 \) such that \( B(x, r) \subseteq Y \).

Since \( B(x, r) \) is open in \( X \) (by Theorem 1.1), \( B(x, r) \subseteq U \).

Thus \( x \in B(x, r) \subseteq U \).

\[ \therefore \text{int } Y \subseteq U. \]

\[ \therefore U \subseteq \text{int } Y. \]

\[ \therefore U \subseteq Y. \]

4. \( \text{let } U' = \{ A : \forall A \subseteq X \text{ and } A = \overline{A} \} \).

Check \( x \in \overline{A} \). Suppose \( B(x, r) \cap Y = \emptyset \) for some \( r > 0 \).

Then \( X - B(x, r) \) is a closed set that contains \( Y \)

\[ \text{i.e., } X - B(x, r) \subseteq \overline{Y} \text{ and } x \in X - B(x, r) \Rightarrow \]

\[ \therefore B(x, r) \subset \overline{Y} \neq \emptyset \text{ for all } r > 0 \text{ and } x \in Y. \]

\[ \therefore \overline{A} \subseteq \overline{Y}. \]

Pick \( x \in \overline{Y} \). Suppose \( x \notin A \) for some \( A \subseteq U \). Then \( x \in X - A \)

which is open in \( X \) and so there exists \( r > 0 \) such that \( B(x, r) \subseteq X - A \). Since \( Y \subseteq A \) it follows that \( B(x, r) \cap Y = \emptyset \).

This implies that \( x \notin Y \Rightarrow \).

We conclude that \( x \in A \) for all \( A \subseteq U \) and so \( \overline{Y} = \bigcap A. \)

\[ \text{Conclusion: } \overline{Y} = \bigcap A. \]
5. *Euclidean*: let \((x,y) \in S\) and let \(r > 0\). Then\((x,y) + \frac{r}{2} (x,y) \subseteq B((x,y), r) \cap (\mathbb{R}^2 - S)\) and so\(B((x,y), r) \nsubseteq S\). It follows that \((x,y) \notin \text{int} S\).

Concl.: \(\text{int} S = \emptyset\).

*Discrete*: let \((x,y) \in S\). Then\(B((x,y), \frac{1}{2}) \subseteq B((x,y)) \subseteq S\) and so \((x,y) \in \text{int} S\). Concl.: \(\text{int} S = S\).

6. Write \(C(x,r) = \{y \in X \mid d(x,y) \leq r\}\). Let \(x \in X - C(x,r)\) and define \(S = d(x,r) - r\).

Claim: \(B(x, S) \subseteq X - C(x,r)\).

Proof: let \(q \in B(x, S)\).

Then \(d(x,q) \leq d(x,q) + d(q,r)\)

\[< d(x,q) + S = d(x,q) + d(x,r) - r.\]

It follows that \(r < d(x,q)\) and so \(q \notin C(x,r)\).

Claim Done.

From this we conclude that \(X - C(x,r)\) is open and so \(C(x,r)\) is closed.

In \(\mathbb{R}^2\) with the Euclidean metric, note that \(B((0,0), 1) = \overline{B}(0,0) = \{x \in \mathbb{R}^2 \mid \|x\| \leq 1\}\), but \(C((0,0), 1) = \mathbb{R}^2\).

Concl.: The closure of an open ball is not necessarily equal to the closed ball of the same radius.
Claim: \( \overline{B(x, r)} = C(x, r) \)

Proof: Given \( y \in \overline{B(x, r)} \), there is a sequence \( \{x_n\} \subseteq \overline{B(x, r)} \) such that \( x_n \to y \). Now \( d_e(x_n, y) = d_e(x_n, y) + d_e(y, y) < r + d_e(x_n, y) \to r + 0 = r \)

It follows that \( d_e(x, y) \leq r \) and so \( y \in C(x, r) \)

\( \therefore \overline{B(x, r)} \subseteq C(x, r) \).

Note that we didn't use any properties of \( d_e \) here; this inclusion holds in any metric space.

We now show that \( \overline{C(x, r)} \subseteq \overline{B(x, r)} \). Since \( \overline{B(x, r)} \subseteq \overline{B(x, r)} \), it suffices to show that \( d_e(C(x, r)) \subseteq \overline{B(x, r)} \). So, choose \( y \in \overline{B(x, r)} \) with \( d_e(x, y) = r \). Define \( x = (x_1, \ldots, x_n) \), \( y = (y_1, \ldots, y_n) \) and define a sequence \( z_n = \frac{1}{n} x + (1 - \frac{1}{n}) y \).

Now \( d_e^2(z_n, y) = \sum \left( \frac{1}{n} x_i + (1 - \frac{1}{n}) y_i - y_i \right)^2 \)
\[= \sum \left( \frac{1}{n} x_i - y_i \right)^2 \]
\[= \frac{1}{n} \sum (x_i - y_i)^2 = \frac{1}{n} d_e(x, y) = \frac{r}{n} \to 0 \]

It follows that \( z_n \to y \).

Also, \( d_e^2(z_n, x) = \sum \left( \frac{1}{n} x_i + (1 - \frac{1}{n}) y_i - x_i \right)^2 \)
\[= (1 - \frac{1}{n})^2 \sum (x_i - y_i)^2 = (1 - \frac{1}{n})^2 d_e(x, y) \]
\[< r \]
and so \( \overline{C(x, r)} \subseteq \overline{B(x, r)} \). It follows that \( y \in \overline{B(x, r)} \) and so \( \overline{B(x, r)} \subseteq \overline{B(x, r)} \).

We have shown \( \overline{B(x, r)} = \overline{C(x, r)} \) in \( (\mathbb{R}^n, d_e) \)
7. \( \to \) Assume that \( U \) is open in \((X,d)\) if and only if \( U \) is open in \((X,p)\) and let \( \{x_n\} \subset X \) with \( \lim_{n \to \infty} d(x_n, x) = 0 \) (i.e., \( \{x_n\} \) is a \( d \)-convergent sequence. We must show that \( \{x_n\} \) is \( p \)-convergent. So, let \( \varepsilon > 0 \). Since \( B_p(x, \varepsilon) \) is open in \((X,d)\), there exists \( r > 0 \) such that \( B_d(x, r) \subseteq B_p(x, \varepsilon) \) (here \( B_p(x, \varepsilon) \) denotes the open ball in \((X,p)\)). Similarly, for \( (B_d(x, r)) \), choose \( N \in \mathbb{N} \) such that \( d(x_n, x) < r \) for all \( x_n \in (X,d) \) and hence \( p(x, x_n) < \varepsilon \) whenever \( n \geq N \). It follows that \( \lim_{n \to \infty} p(x, x_n) = 0 \) and so \( \{x_n\} \) converges to \( x \) in \((X,p)\).

\( \leftarrow \) Assume that a sequence \( \{x_n\} \subset X \) converges in \((X,d)\) if and only if it converges in \((X,p)\).

Let \( U \subset X \) be open in \((X,d)\). Then \( X \setminus U \) is closed in \((X,d)\).

Pick \( z \in X \setminus U \). Since \( z \) is adherent to \( X \setminus U \) in \((X,d)\), there exists a sequence \( \{z_n\} \subset X \setminus U \) that converges to \( z \) in \((X,d)\).

By assumption, \( \{z_n\} \) converges to \( z \) in \((X,p)\) and so \( z \) is adherent to \( X \setminus U \) in \((X,p)\). It follows that \( X \setminus U \) is closed in \((X,p)\) and so \( U \) is open in \((X,p)\). * See next page.

8. Assume \( \sum d(x_n, x_{n+1}) = S \). Let \( \varepsilon > 0 \) be given. There exist \( N \) such that if \( p > N \), then \( S - \sum d(x_k, x_{k+1}) < \varepsilon \). Now, let \( m, n \geq N \) and assume without loss of generality \( m < n \). Then
\[
d(x_m, x_n) \leq d(x_m, x_{m+1}) + \ldots + d(x_{n-1}, x_n) \leq \sum_{k=p}^{N-1} d(x_k, x_{k+1}) < \varepsilon.
\]

\( \therefore \) \( \{x_n\} \) is Cauchy.
In #7 (e) I used the fact that if \( \mathbf{z} \to z \) in \((X, d)\) and \( \mathbf{z'} \to z' \) in \((X', f)\), then \( z = z' \). You should confirm that this is indeed true.

8. (a) (Indirect) Let \( S_n = \sum d(x_k, x_{k+1}) \). Since \( \sum d(x_k, x_{k+1}) \) is convergent, the sequence \( S_n \) also converges to some number \( S \) in \( \mathbb{R} \). It follows that \( S_n < R \) is Cauchy.

So, given \( \varepsilon > 0 \) let \( N \) be such that if \( m, n \geq N \), then \( |S_{m-1} - S_{n-1}| < \varepsilon \). Now, if \( m = n \geq N \) we have

\[
d(x_m, x_n) = d(x_m, x_{m+1}) + \cdots + d(x_{m-1}, x_n) = S_{m-1} - S_{n-1} < \varepsilon
\]

\[\therefore \mathbf{z}_n \text{ is Cauchy.}\]