Nature of Modern Math

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March 8, 2015
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A Note to the Student

There are few mathematical prerequisites for reading this book, it is almost entirely self contained. A certain depth and breadth of character, on the other hand, are absolutely essential.

Chief among the necessary attributes are:

1. **Passion.** If you are not passionate about learning, if you view your education merely as a collection of hurdles to clear and grades to post, you have come to the wrong place. Without passion, the virtues that follow are a train with no engine.

2. **Generosity.** Think of this book as the itinerary for a great trek and the class as the expedition party. The trail will be hard going at times, but we will fail and succeed as a group. Prepare to give of your time, your insight, and your passion so that we all may receive.


4. **Determination.** We will get stuck. We will flounder. Greet every doubt and every obstacle with a smile and fresh resolve. We are determined to triumph.

5. **Accountability.** You are responsible for your own education. The opportunity to change how you see and do mathematics is placed here before you, commit to seizing it. Discere faciendo.

6. **Optimism.** The slate is wiped clean. Whatever your past mathematical successes and failures, focus now on the present. Believe that it is possible for one class to forever change the course of your mathematical life. Believe that this is that class.

This book is an attempt to orchestrate and present conditions suitable for discovery. Discovery of new mathematical ideas, discovery of interconnections between them, and discovery of one’s own mathematical voice.
Because no one can discover something for you, nearly every structural object in this text requires action.

**Definition.** Definitions are the key objects in this book. They are agreements upon the meaning of terms or phrases. The defined term or phrase is always in emphasized text. These agreements are inviolable and no defined term is ever used loosely. The required actions are memorization, digestion and contextualization.

**Example.** Examples are provided to highlight or lend interest to a new term or idea and generate discussion. Required actions are digestion and contextualization.

**Exercise.** Exercises are explicit actions requested of the reader, usually in order to put new terms or ideas to use, test comprehension, and generate discussion.

**Proposition.** Propositions are statements that may or may not be true. They depend only on definitions, propositions and theorems that have come before them in the book. The first required action is to assess whether the statement is true or false. The remaining required action depends on your assessment. If true, prove it; if false, provide a counterexample to the statement.

**Theorem.** Theorems are statements that are true and are valuable because of wide applicability or deep consequence. They depend only on definitions, propositions and theorems that have come before them in the book. The required action is to prove the theorem.

This book is not typical. From these pages you will get only what you give. Read with conviction and energy. Have pencil and paper at hand. Take no statement for granted and maintain a healthy skepticism.

This book contains all that you will need and nothing you won’t. Do not consult other texts or the internet. In this class, earnest failure outweighs counterfeit success; you need not feel pressure to hunt for solutions outside your own creative and intellectual reserves.

This book is the open road and a full tank of gas.

This book was written for you.

_San Luis Obispo, 2015_
Chapter 1

The Words Matter

If you don’t say it well, people won’t know what you’re talking about.

1.1 Wordplay

We begin with a definition.

Definition 1.1. The length of a word is however many letters long it is. Given a word \( W \), denote its length by \( l(W) \).

Example 1.2. \( l(\text{book}) = 4 \) and \( l(\text{trigonometry}) = 12 \). \( \Box \)

We would next like to develop a measure of similarity between two words of equal length.

Definition 1.3. The difference between two words of equal length is the number of letter positions in which the two words have different letters. Given two equal length words \( W \) and \( V \), denote their difference by \( d(W,V) \).

Example 1.4. \( d(\text{book}, \text{look}) = 1 \) because they have different letters in exactly one position (the first position on this example). On the other hand, \( d(\text{cat}, \text{hog}) = 3 \) because they have different letters in all three letter positions. \( \Box \)

Definition 1.5. A list of distinct words is called stable if all the words on the list are the same length and the difference between consecutive words is equal to 1.

With these few definitions in hand, our “theory of words” will now blossom. As you consider the following propositions, remember that their content is not the real prize; you are unlikely to encounter any of these results in your mathematical lives.

beyond this course. The real prize is your own burgeoning awareness that theories are built on definitions and that deep and precise understanding of the definitions is one of the key prerequisites for mathematical success.

Determine whether each proposition is true or false. Justify your conclusion thoroughly.

**Proposition 1.6.** There exists a stable list of five length-4 words.

**Proposition 1.7.** Every stable list of length-2 words has fewer than 1000 words.

**Proposition 1.8.** If $W$ and $V$ are on the same stable list, then $d(W, V) < l(W)$.

As we define new notions into an existing theory, the theory grows both in richness and complexity.

**Definition 1.9.** A list of distinct words is called tight if all the words on the list are the same length and $d(W, V) \leq 2$ for any two words $W$ and $V$ on the list.

Determine whether each proposition is true or false. Justify your conclusion thoroughly.

**Proposition 1.10.** There exists a fourth word $W$ beginning with an “h” such that the list

\[
\{ \text{sit, hut, sum, } W \}
\]

is tight.

**Proposition 1.11.** If the list

\[
\{ \text{sit, hut, sum, } W \}
\]

is tight and $W$ begins neither with “s” nor “h”, then the last letter of $W$ is a “t”.

It is natural to wonder about the relationships, if any, between tight and stable lists.

**Proposition 1.12.** There exist tight stable lists of length-3 words.

**Proposition 1.13.** All tight lists are stable.

**Proposition 1.14.** All stable lists are tight.
1.2 Board Games

Chess boards of various sizes are fertile ground for some interesting mathematics. In this section we visit just a few of these ideas.

1.2.1 Bizzarro Tic-Tac-Toe

Exercise 1.15. Draw a four-by-four chess board, complete with shading on every other square.

Bizzarro Tic-Tac-Toe is a two player game played on the four-by-four chess board created above. In this game, player $A$ places pennies on any two squares. Player $B$ then places paper clips on the board so that each clip covers two unoccupied squares and no clips overlap. To win, player $B$ has to lay down seven paper clips; otherwise player $A$ wins.

Exercise 1.16. Get somebody to play you at Bizzarro Tic-Tac-Toe and record the final configurations of four distinct games.

Did $A$ always win? Did $B$? Is there a winning strategy for $A$? To start to get at the answer, the following proposition concerns whether there is some “color parity” to $B$’s play.

Proposition 1.17. However many clips $B$ ends up playing, the clips cover equal numbers of squares of each color.

Based on your determination of the truth or falsity of Proposition 1.17, we arrive at our first theorem:

Theorem 1.18. Player $A$ has a winning strategy for Bizzarro Tic-Tac-Toe.

1.2.2 Chesstris

We next consider a version of Tetris played on a certain chess board.

Exercise 1.19. Draw a four-by-five chess board, complete with shading on every other square.

Chesstris is a one player game played on the four-by-five chess board created above.
CHAPTER 1. THE WORDS MATTER

Exercise 1.20. Draw a four by one “Tetris” piece and label it $A$.

Exercise 1.21. Draw a two by two square “Tetris” piece and label it $B$.

Exercise 1.22. Draw a “Tetris” piece in the shape of a “T” three squares across the top and one square below the middle top square and label it $C$.

Exercise 1.23. Draw a “Tetris” piece in the shape of an “L” with height three squares and base two squares and label it $D$.

Exercise 1.24. Draw a “Tetris” piece in the shape of an “S” by shifting the top two squares of $B$ one square to the right and label it $E$.

The object of Chesstris is to arrange the five pieces $A, B, C, D, E$ on the board without overlapping.

Exercise 1.25. Play some Chesstris!

Proposition 1.26. If pieces $A, B, D, E$ are played, then the remaining squares on the board are two of each color.

Based on your determination of the truth or falsity of Proposition 1.26, we arrive at our next theorem:

Theorem 1.27. Chesstris cannot be won.

1.2.3 Truces

Definition 1.28. A queen is a chess piece that can attack along rows, columns and diagonals. A truce is an arrangement of mutually non-attacking queens.

Proposition 1.29. There exists a truce of three queens on the three-by-three chess board.

Proposition 1.30. There exists a truce of eight queens on the standard eight-by-eight chess board.

Proposition 1.31. There does not exist a truce of six queens on a six-by-six chess board in which some queen is in a corner of the board.

Proposition 1.32. There exists a truce of six queens on a six-by-six chess board.
1.3 Mixing Metaphors

It has been said that good mathematicians see analogies between theories, whilst
great mathematicians see analogies between analogies. In this final section of chapter
one, we consider three problems and attempt to see the common structure that binds
them.

1.3.1 Tea & Wine

This subsection concerns a math problem disguised as an old puzzle.

Exercise 1.33. Draw a cup of tea beside a barrel of wine.

In this puzzle, you take a spoonful of tea and pour it into the barrel of wine, mix
thoroughly, and then take a spoonful of the mixture and pour it back into the
cup of tea.

Proposition 1.34. The amount of tea in the wine barrel equals the amount of wine
in the teacup.

1.3.2 Coin Flips

This subsection concerns a math problem disguised as an old parlor trick.

Dump a bunch of coins on the table, some heads up, some tails up. Look away
and ask your neighbor to count the number of heads and then pull aside that many
coins however they like. Tell them not to show you their pulled aside collection.
Now ask your neighbor to flip over all the coins they pulled aside.

Exercise 1.35. Without looking at their collection, theatrically announce the num-
ber of heads up coins your neighbor has showing!

Proposition 1.36. This trick works every time.

1.3.3 Trap Madness

This subsection concerns a geometry problem.

Definition 1.37. A trapezoid is a four-sided figure with two sides parallel to one
another.
Exercise 1.38. Draw a trapezoid with one of the parallel sides as the base of the figure.

Exercise 1.39. Draw in the diagonals connecting opposite corners.

Proposition 1.40. *The two triangles that don’t share a side with the base or top of the figure are of equal area.*
Chapter 2

The Mod Squad

You like your credit card to be secure, right?

2.1 The Integers

An eye-opening feature of mathematics is that sometimes there are “alternate realities” in which addition, subtraction, multiplication, and division are different but not any better or worse than the ones we’re used to. In this chapter we explore just such an alternate reality and see what it may be good for in real life.

We begin with a definition.

Definition 2.1. The integers $\mathbb{Z}$ are the set of all negative and nonnegative whole numbers:

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}.$$

Example 2.2. 18 and -112 are integers, but $2/3$ and $\pi$ are not. □

We have all said things like “six is divisible by two”; the next definition makes statements like this precise.

Definition 2.3. We say that the integer $a$ divides the integer $b$ if there exists an integer $k$ such that $b = ak$.

Example 2.4. The integer 6 divides the integer 42 because $42 = 6 \cdot 7$. The integer 8 does not divide the integer 34 because there is no integer $k$ such that $34 = 8k$. □

Proposition 2.5. Let $a$, $b$ and $c$ be integers. If $a$ divides $b$, then $a$ divides $bc$. 
Proposition 2.6. Let $a$, $b$ and $c$ be integers. If $a$ divides $b$ and $a$ divides $b + c$, then $a$ divides $3c$.

Exercise 2.7. Create a definition for what it means for an integer to be even.

Exercise 2.8. Create a definition for what it means for an integer to be odd.

Proposition 2.9. If $x$ and $y$ are even integers, then $x + y$ is an even integer.

Proposition 2.10. If $x$ and $y$ are odd integers, then $x + y$ is an even integer.

Proposition 2.11. If $x$ and $y$ are even integers, then 4 divides $xy$.

Proposition 2.12. If $x$ is an integer, then $x^2 + x + 3$ is an odd integer.

Theorem 2.13. The product of consecutive integers is an even integer.

2.2 Modular Arithmetic

The integers are familiar to us. They alone don’t constitute an “alternate” mathematical reality. But if we introduce a new notion of “equals” and corresponding new notions of arithmetic, we all of a sudden find ourselves in a new mathematical place. What do we do in a new place? We explore!

Definition 2.14. Let $m$ be an integer with $m \neq 0$. Given integers $x$ and $y$, we write

$$x \equiv y \pmod{m}$$

if $m$ divides $x - y$. The symbols $x \equiv y \pmod{m}$ are read “$x$ is congruent to $y$ modulo $m$.”

Example 2.15. It is true that $9 \equiv 2 \pmod{7}$ because 7 does divide 9-2; but $9 \not\equiv 2 \pmod{5}$ because 5 does not divide 9-2. □

So, here we are in a totally different world. If we take $\equiv$ as some sort of “alternate” version of good old $=$ (“equals”), then in this new world the integers 9 and 2 aren’t different at all (mod 7 at least)!! Now, if we are to do solid mathematics in this new world, we’ll need to get our bearings and learn how to do arithmetic here. So it’s back to square one, let’s learn to add and multiply...
This first proposition says that adding in the new world amounts to adding
the usual way.

**Proposition 2.16.** Let \( m \) be an integer. If \( a + b = c \), then \( a + b \equiv c \) (mod \( m \)).

The next one says the same for multiplication.

**Proposition 2.17.** Let \( m \) be an integer. If \( a \cdot b = c \), then \( a \cdot b \equiv c \) (mod \( m \)).

The next few theorems are all about adding and multiplying in the new “mod
world”. Stick to the definitions to prove the theorems!

**Theorem 2.18.** Let \( m \) be an integer. If \( k \) is an integer and \( a \equiv b \) (mod \( m \)), then
\[ a + k \equiv b + k \] (mod \( m \)).

**Theorem 2.19.** Let \( m \) be an integer. If \( k \) is an integer and \( a \equiv b \) (mod \( m \)), then
\[ ka \equiv kb \] (mod \( m \)).

**Theorem 2.20.** Let \( m \) be an integer. If \( a \equiv b \) (mod \( m \)) and \( c \equiv d \) (mod \( m \)), then
\[ a + c \equiv (b + d) \] (mod \( m \)).

**Theorem 2.21.** Let \( m \) be an integer. If \( a \equiv b \) (mod \( m \)) and \( c \equiv d \) (mod \( m \)), then
\[ ac \equiv (bd) \] (mod \( m \)).

### 2.3 What is Mod, Really

In the previous section, we learned a bunch of facts about modular arithmetic, but
it can seem like “just symbol pushing.” Fair enough! In this section we begin to get
a better feel for what modular arithmetic is all about, beyond the symbols.

**Exercise 2.22.** The integer 5 does not divide the integer 19. What is the remainder
when we divide 19 by 5?

**Exercise 2.23.** Draw a long number line with the integers from -20 to 20 on it.
Put a box around the integers \{0, 1, 2, 3, 4\} and “clump” the remaining integers in
successive groups of five. What do you observe about where the integer 19 sits in
its “clump.” Is there a connection to Exercise 2.22?

**Proposition 2.24.** If \( x \equiv 4 \) (mod 5), then there is an integer \( k \) such that \( x = 5k + 4 \).

The items above suggest that congruence mod \( m \) has to do with remainders
upon division by \( m \). Now, we don’t need modular theory to tell us that the remain-
der is 4 when we divide 19 by 5; we’ve known this since elementary school. But
larger numbers may pose a greater challenge that modular congruence could help
us overcome.
Exercise 2.25. What theorem in Section 2.2 justifies the statement $112 \cdot 65 \equiv 32 \cdot 57 \pmod{8}$?

Looking at the proposition above, without modular theory it might not be so easy to see that $112 \cdot 65$ and $32 \cdot 57$ have the same remainder upon division by 8.

Proposition 2.26. $112^3 \equiv 32^3 \pmod{8}$.

Proposition 2.27. The last digit of $17^{17}$ is a 7.

Exercise 2.28. What time will it be 1000 hours after class ends on Friday?

2.4 Check Digits

There are three settings from daily life in which modular arithmetic plays a quiet but key role. In this section, we investigate these practices one at a time and conclude that modular arithmetic can be useful in the right hands.

2.4.1 Proof of Purchase

Example 2.29. Get out an item you recently purchased (cereal, gatorade, spiral notebook, whatever’s handy) and locate the bar code. See Figure 2.1 for an example.

![Figure 2.1: A standard UPC code.](image)

Those digits you see under the bar code (all twelve of them, even those tiny ones in either corner) are not random!

Exercise 2.30. Working from right to left, multiply the first, third, fifth, seventh, ninth and eleventh digits by 1 and the second, fourth, sixth, eighth, tenth and twelfth digits by 3. Now add all those results; what did you get?
Exercise 2.31. Repeat Exercise 2.30 with another barcode. What did you get?

Hopefully your answers to the previous exercises both end in zero. Put another way, your answers are both divisible by ten. Put into our language, both answers are congruent to zero (mod 10). This is true of every UPC code! The rightmost digit is called a check digit and must be correct in order for the UPC to be validated at the register.

Exercise 2.32. What must the check digit \( y \) be in order for the UPC 7-96714-78601-\( y \) to be valid?

Probably the most common error in reading or writing long strings of letters is transposition of consecutive digits. For example, the code 0-28400-07056-0 is valid but the code 0-24800-07056-0 is not. There is only one problem: sometimes transposing digits doesn’t result in an invalid code.

Proposition 2.33. If two adjacent digits have a difference of five, then transposition of those two digits goes undetected in the UPC system.

2.4.2 ISBN Numbers

Example 2.34. Get out a book and locate the ISBN number. See Figure 2.2 for an example.

Figure 2.2: A standard ISBN number.

Those digits you see near the bar code (there are ten in this case, with some hyphens) are not random! Here the check digit is calculated differently than with UPC codes.
**Exercise 2.35.** Working from *left to right*, multiply the first digit by 10, the second digit by 9, and so on until you multiply the ninth digit by 2. Now add all those results plus the tenth digit (the *check digit*); what is your answer congruent to modulo 11?

**Exercise 2.36.** Repeat Exercise 2.35 with another book. What did you get?

Hopefully your answers to the previous exercises were both zero. This is true of every ISBN number! The rightmost check digit must be correct in order for the ISBN to be valid.

**Exercise 2.37.** What must the check digit \( y \) be in order for the ISBN 0-14-044054-y to be valid?

As mentioned in Subsection 2.4.2, probably the most common error in reading or writing long strings of letters is *transposition* of consecutive digits. One problem with the UPC system was that sometimes transposing digits doesn’t result in an invalid code. Does the ISBN system suffer from the same malady?

**Proposition 2.38.** No transposition of two digits goes undetected in the ISBN system.

### 2.4.3 Credit Card Numbers

**Example 2.39.** Get out a credit card. See Figure 2.3 for an example.

![Figure 2.3: A standard credit card.](image)

The digits of the credit card number are not random! Here the check digit is calculated differently than in the UPC and ISBN systems.

The process used to determine the check digit is called the Luhn algorithm (mod 10), named after IBM scientist Hans Peter Luhn. This algorithm works as follows:
(1) Begin by doubling all even-positioned digits when counting from right to left. If doubling results in a two-digit number, add the digits. For instance, if the original digit were a 6, doubling it would give 12, so use \(1 + 2 = 3\).

(2) Determine the sum of the results from Step (1) and each of the unaffected (odd-positioned) digits in the original number.

(3) Verify the account number by determining if the sum from Step (2) is congruent to 0 (mod 10).

**Exercise 2.40.** Apply the Luhn algorithm to determine if the credit card account number 5314 7726 8593 2112 is valid.

**Exercise 2.41.** What must the check digit \(y\) be in order for the account number 6011 4387 1005 123\(y\) to be valid?

How does the Luhn algorithm do detecting transpositions?

**Proposition 2.42.** There are two digits whose transposition goes undetected by the Luhn algorithm.
Chapter 3

A Bridge Too Far?

I am the master of my fate: I am the captain of my soul.

From Invictus, by William Ernest Henley

3.1 The Bridges of Konigsberg

There is a famous old problem about the city of Konigsberg in Germany. Situated on the banks of the Pregel river, Konigsberg was a thriving and important city for trade and travel. In the middle of the river there is an island called Kneiphof, literally meaning “pub yard”, and the citizens of Konigsberg built seven bridges for easy access to Kneiphof. See Figure 3.1 below.

Figure 3.1: The seven bridges of Konigsberg.
The citizens of Konigsberg were delighted with their bridges and began to wonder whether it was possible to walk around the city crossing each bridge exactly once. Though no one seemed able to pull off the trick, neither was anyone able to prove it couldn’t be done. So the last question of the quarter is: what is the solution of the Konigsberg bridge problem?

3.2 Euler Paths

To start to build the machinery we’ll need to solve the Konigsberg bridge problem, we need to consider so-called graphs and circuits. As always, we start with a definition.

**Definition 3.1.** A *network* is a figure consisting of points, called *vertices*, connected by lines called *edges*.

![Some networks](image)

**Figure 3.2: Some networks.**

**Definition 3.2.** The *degree* of a vertex in a network is the number of edges that meet at it.

**Exercise 3.3.** Find the degree of each vertex of each network in Figure 3.2.

**Definition 3.4.** An *Euler path* (it’s pronounced “Oiler”) is a path on a network that traces each edge exactly once.

Informally, a network has an Euler path if we can draw it without retracing any edge or lifting our pencil. Some networks have this feature and some don’t. Deciding which are which is our challenge.

**Proposition 3.5.** *The top left graph in Figure 3.2 has an Euler path.*
Proposition 3.6. *Seven of the eight networks in Figure 3.2 have Euler paths.*

In light of Proposition 3.6, it sure would be nice to know in advance which of those networks would have an Euler path.

**Exercise 3.7.** Use your answers to Exercise 3.3 and Proposition 3.6 to guess a condition on a network that tells whether or not it has an Euler path.

### 3.3 Degree is the Key

Whatever you may have guessed in Exercise 3.7, the degrees of the vertices surely had some bearing. In this section, we formalize our thoughts on the existence of Euler paths by more carefully considering the various possible degrees of vertices. To do it well, we introduce some terminology.

**Definition 3.8.** The *initial vertex* of an Euler path is where the path starts and the *terminal* vertex is where the path ends.

**Exercise 3.9.** Draw an Euler path of five vertices with initial and terminal vertex at the same vertex.

**Exercise 3.10.** Draw an Euler path of five vertices with initial and terminal vertices at different vertices.

**Exercise 3.11.** Draw an Euler path of five vertices with terminal vertex of degree one.

In order to formulate our discussion of the existence of Euler paths, it may also be useful to consider vertices that are “next to each other.” The following definition formalizes this notion.

**Definition 3.12.** Two vertices of a network are called *adjacent* if there is an edge of the network connecting them.

**Proposition 3.13.** If $v$ is a vertex adjacent to the terminal vertex of an Euler path and the degree of $v$ equals 1, then $v$ is the initial point of the path.

We are now ready to prove our assertion in Exercise 3.7. Whatever the degree of a vertex $v$ in an Euler path, some of the edges that meet at $v$ were traveled into $v$ while others were traveled out of $v$.

**Proposition 3.14.** If $v$ is not the terminal vertex of an Euler path, then there are at least as many edges out of $v$ as there are into $v.$
**Proposition 3.15.** If \( v \) is not the initial vertex of an Euler path, then there are at least as many edges into \( v \) as there are out of \( v \).

Taken together, Propositions 3.14 and 3.15 establish the following theorem.

**Theorem 3.16.** If the degree of the vertex \( v \) in an Euler path is odd, then \( v \) must either be the terminal or initial vertex of the path.

As a corollary to Theorem 3.28, a network with an Euler path can only have so many vertices of odd degree.

**Exercise 3.17.** Formulate and prove a theorem of the form:

\[
\text{Theorem: If a network } N \text{ has more than } ?? \text{ vertices of odd degree, then } N \text{ does not possess an Euler path.}
\]

### 3.4 Full Circle

With our thoughts concerning degrees of vertices fully fledged, we now return to the question that started the whole discussion. Recall the town of Konigsberg with its seven bridges depicted below:

![The seven bridges of Konigsberg.](image)

**Figure 3.3:** The seven bridges of Konigsberg.

We are now prepared to answer the question of whether one can walk through the town crossing each bridge exactly once. The idea is to imagine each of the four connected land masses as shrinking down to the point that only one person at a time can stand on them.
Exercise 3.18. Redraw Figure 3.3 with the four land masses as mere dots connected by the bridges. You should have a network with four vertices and seven edges.

Now look at the network you drew in Exercise 3.18 and determine the degree of each vertex.

Theorem 3.19. There is no walk through the seven bridges of Konigsberg that crosses each bridge exactly once.

Success! Here was a prime example of taking a question that seemed non-mathematical and solving it by putting it into a mathematical context. The story is not over, of course (maybe that’s the true nature of modern math: the story is never over).

Exercise 3.20. Add a single bridge to Figure 3.3 so that one can walk the eight bridges of Konigsberg crossing each bridge exactly once. Draw the corresponding network and Euler path.

Now that we know what to check on a bridge problem, we can consider harder bridge problems that would have been mind-bogglingly impossible before Math 112. Consider the following town of Pretendberg:

Figure 3.4: The fifteen bridges of Pretendberg.

Proposition 3.21. It is possible to walk the town of Pretendberg crossing each bridge exactly once.
CHAPTER 3. A BRIDGE TOO FAR?

3.5 Complete Graphs

As a final foray into the study of networks, we explore another tantalizing old problem. As you may expect at this point, we begin with a definition.

Definition 3.22. A complete graph is a network in which every vertex is adjacent to every other vertex. The complete graph with \( n \) vertices is denoted by \( K_n \).

Exercise 3.23. Draw \( K_3 \), \( K_4 \) and \( K_5 \).

Exercise 3.24. Formulate and prove a theorem of the form:

Theorem: \( K_n \) has exactly ?? edges.

Clearly some of the \( K_n \) possess Euler paths and some do not.

Exercise 3.25. Determine all the values of \( n \), with justification, such that \( K_n \) possesses an Euler path.

Now that the question of Euler paths is settled for complete graphs, we turn our attention to the final question of the quarter. Imagine coloring the edges of \( K_n \) either red or blue. Depending on how we color all those edges, there may or may not appear a monochromatic triangle (a triangle where all the edges are the same color).

Exercise 3.26. Draw a coloring of \( K_4 \) that does have a monochromatic triangle and a coloring of \( K_4 \) that doesn’t have a monochromatic triangle.

Exercise 3.26 shows that a coloring of \( K_4 \) may or not possess a monochromatic triangle. We are led to wonder if a monochromatic triangle is ever guaranteed for any \( K_n \). Put another way, is there a big enough value of \( n \) such that every coloring of \( K_n \) has a monochromatic triangle? We know \( n = 4 \) isn’t big enough, but how about \( n = 5 \)?

Proposition 3.27. There is a coloring of \( K_5 \) that lacks a monochromatic triangle.

Whatever we decide about Proposition 3.27, the situation is well understood of \( K_6 \):

Theorem 3.28. Every coloring of \( K_6 \) possesses a monochromatic triangle.

Corollary 3.29. At any party of six people, there will either be a group of three mutual friends or a group of three mutual strangers.
Though we are at the end of our journey, we leave off at just the beginning of the story of complete graphs. As it turns out, questions like Corollary 3.29 are extremely difficult. To wit, if we ask for groups of four mutual friends or four mutual strangers (rather than groups of three), we aren’t guaranteed success unless we invite 18 people to the party. More unbelievable still, if we ask for cliques of five people... wait for it... NOBODY KNOWS!

At this very moment, mathematicians are working on this very problem and the best anybody knows is that we need between 43 and 49 people at a party in order to guarantee six mutual friends or mutual strangers. Why not just check parties of 43, 44, 45, 46, 47, 48 and 49 to see which it is? Can you imagine the number of different colorings of $K_{43}$ that we’d need to check for monochromatic pentagons? It’s unfathomably many, and no supercomputer running from now until the sun burns out could check them all!

In many ways that’s the story of math. Solving tantalizing real problems, not by clumsy brute force, but by elegant application of reason alone. There is poetry in this, and you are now invited to read along.
Bibliography


