9.0 Kinematics of rigid bodies—velocity

As explained in the beginning of this book, this study of Dynamics is divided into the study of particle motion (point masses) and then the study of bodies extended in space or rigid bodies. Each of these is subdivided into the study kinematics—pure motion without regard to forces involved in causing the motion or resulting from the motion—and the study of kinetics, where the interplay of forces and moments and the motion is considered. Thus we are at the third quarter of this study and consider now the pure motion, without forces, of extended, rigid bodies.

9.1 Rigid bodies vs. particles

Rigid bodies are physical objects that are extended in space. Particles, as we have seen, are point masses. That is, they are objects whose physical extension in space is unimportant to the analysis at hand. The size of an object is often not important. For example, the mechanics of the planets in the solar system revolving around the sun can be analyzed using the simplification that the masses of the planets and the sun are concentrated at a single point, even though planets and the sun are quite large bodies. It’s the scale of the overall problem that determines whether or not reducing masses to single points is possible.

With extended objects it is possible to have two or more points on a body. The figure below shows the standard 2-D blob that is often used to discuss rigid-body motion. The blob contains two points A and B. It is possible to look at the motion of A, the motion of B, the motion of A viewed by an observer on B, or the motion of point B viewed by an observer at A. Since the body is rigid, the straight-line distance between A and B does not change. That is A does not approach B nor does it recede from B. The same is true, of course, of B, regarded from the standpoint of A. So an observer standing on A can only see B going sideways either toward the left or right.

![Figure 9.1 – Relative motion of two points on a rigid body](image)

9.2 Types of planar motion

A rigid body undergoes two basic types of motion. It can translate—that is, move without rotating—and it can rotate. Translation can be movement in a straight line, or it can be movement along a curvilinear path. What characterizes translation is that a line scored on the surface of a body does not change its orientation as it moves. The figure below shows two cases of translation—rectilinear translation (along
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A straight-line path) and curvilinear translation. In both cases, the line between points \(A\) and \(B\) stays pointed in the same direction as the body moves.

![Rectilinear translation](image1.png)  
(a) Rectilinear translation

![Curvilinear translation](image2.png)  
(b) Curvilinear translation

**Figure 9.2 – (a) Rectilinear and (b) curvilinear translation**

The other type of motion is rotation. *Pure rotation* is rotation about the body’s mass center. In this case one can say that the body doesn’t go anywhere since the center of mass is stationary. Rotation can also take place around a fixed point that is not the mass center of the body. This is called *fixed-axis rotation*.

![Pure rotation](image3.png)  
(a) Pure rotation

![Fixed-axis rotation](image4.png)  
(b) Fixed-axis rotation

**Figure 9.3 – Rotation**

A body can also undergo translation + rotation—that is, a combination of both types of motion, occurring simultaneously, and with no fixed axis of rotation. Rather the point about which the object rotates changes. This is called *general plane motion*.

![General plane motion](image5.png)

**Figure 9.4 – General plane motion**
Rotation is what distinguishes rigid-body motion from particle motion. Particles may move in a circular path about some point, but to talk of the rotation of the particle itself or of its orientation makes no sense. For a rigid body, it undergoes rotation if any line scribed on it (such as AB above) changes its orientation. The body may move along a complicated path, as in Figure 9.2(b) above. But if a line scribed on it does not change its orientation in this motion, there is no rotation involved.

9.3 Relative motion

When studying rigid-body motion, the notion of an observer on a point on a body observing the motion of another point on the rigid body is useful. Take an observer on point B looking at point A, as shown in Figure 9.5. Point A neither comes toward B nor does it recede from B, because the body is rigid. The observer is not aware of his/her own motion. In translation, all points on a body have the same motion. So to an observer on B regarding A, it seems there is no motion at all. Thus the relative motion between A and B is 0.

The only motion of A that an observer on B sees is rotation. Since the observer at B is unaware of his/her own motion, it looks to him/her that the body is rotating about his/her observation point. Every point on the body looks as if it is rotating about B. To an observer at A, it looks as if the entire body is rotating about A. Since no point on the rigid body can come toward nor recede from the observer, all the observer sees is sideways motion—to the left or the right. This is simply a result of the body’s rotation. Counter-clockwise rotation looks like movement to the left (as shown above); clockwise rotation looks like movement to the right.

This relative motion takes on the characteristics of rotation about a point. For example the apparent sideways speed of A to an observer at B depends on the angular speed, \( \omega \), of the body, and the distance \( r_{AB} \) between the two points. With vectors

\[
\vec{v}_{A/B} = \vec{\omega}_{AB} \times \vec{r}_{A/B}
\]
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This is a very important relationship in rigid-body kinematics. The subscripts and how to interpret them must be understood very well. First, “A/B” means “A with respect to B”, that is “the movement of point A to an observer riding along on B”. This is different than “B/A”. In fact it is its opposite. “B/A” means that the observer is on A looking at B. If an observer on B looking at A sees A going to his/her left, as shown above, this means that the body is rotating counter-clockwise. The speed is $\omega_{AB}r_{AB}$, where $r_{AB}$ just denotes the distance between A and B without specifying a reference point. To an observer on B looking at the motion of A, this same rotation makes it look as if A is going to the left. Note that two velocity vectors $\vec{v}_{A/B}$ and $\vec{v}_{B/A}$ have the same speed, $\omega_{AB}r_{AB}$, but they are pointed in opposite directions.

Note that the subscript of $\vec{v}_{A/B}$ is simply AB, not A/B nor B/A. This intentional and is a very important point. A rigid body has only one angular velocity. There is no such thing as $\omega_{A/B}$ nor $\omega_{B/A}$. They make no sense. The body rotates. End of story. It does not rotate differently depending on a specific reference point. The subscript “AB” simply denotes the body that points A and B are on. So “AB” means the same thing as “BA”, though in an analysis it would be confusing to switch the order of these letters willy-nilly. In this book, we shall always write these non-observer-referencing subscripts with the letters in alphabetical order.

So in summary the equation

$$\vec{v}_{A/B} = \vec{\omega}_{AB} \times \vec{r}_{A/B}$$

must have the following features:

- The subscript of $\vec{r}$ must be the same as the subscript of $\vec{v}$ with the same letters in the same order.
- $\vec{\omega}$ will have a non-observer-referencing subscript, preferably with the letters in alphabetical order.

9.4 Vector approach to rigid-body kinematic analysis of velocities

There are several approaches to analyzing 2-D rigid-body kinematic problems. They can be solved simply trigonometrically, for example, applying the rules of trigonometry to a mechanism, noting geometric features of it. Another approach is a more formal, rigid vector approach. This approach is explained in this section. For beginners in rigid-body motion, it offers a structure that is set and can be applied to all 2-D kinematic problems involving rigid bodies.

The key to this approach is to pick two points on a rigid body whose velocities are known or are partially known. In many cases the magnitude of the velocity, the speed, of a point is not known, but it is constrained to move in a certain direction. Let’s say that there is a rigid body with two points on it, A and B, about which the velocities are known or partially known. The relative velocity relationship can be written relating the velocities of these two points.

$$\vec{v}_B = \vec{v}_A + \vec{v}_{B/A}$$
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Note that the subscripts in this equation follow the pattern $B-A-B-A$. This is always the case. It would be equally valid to write the $A-B-A-B$ equation, and in most problems it does not matter which one is written. Both are valid. From this equation, we can expand the term $\vec{v}_{A/B}$.

$$\vec{v}_{A/B} = \vec{\omega}_{AB} \times \vec{r}_{A/B}$$

These two equations form the basis of this vector approach and are employed sequentially over and over as we work our way through the mechanism, applying this equation in stages, to pairs of points on different components of the mechanism. To show how this is done, let’s look at a classical mechanism in rigid-body kinematics.

Example 9.1 - Slider/Crank mechanism

At right is shown a slider/crank mechanism, which is the basis for converting linear motion into circular motion in a reciprocating engine or compressor. The piston $B$ runs vertically up and down in the cylinder. The link $AB$ is the connecting rod. $OA$ is the crank or crank arm. It rotates about the fixed crank center at $O$. For this discussion let’s assume that $\omega_{OA}$ is constant and is given. The lengths of the links—$l_{OA}$ and $l_{AB}$—are also given, as are $\theta$ and $\phi$ in this position. We are interested in the angular velocity of $AB$ and the speed of the piston in this configuration.

We start with what is known ($\vec{\omega}_{OA}$) and work our way toward what is unknown ($\vec{v}_B$).

We can get the velocity of $A$, knowing $\vec{\omega}_{OA}$.

$$\vec{v}_A = \vec{\omega}_{OA} \times \vec{r}_{A/O}$$

The rotational velocity of $OA$ is given ($\vec{\omega}_{OA} = \omega_{OA}\hat{k}$), and $\vec{r}_{A/O} = l_{OA}(\cos \theta \hat{i} + \sin \theta \hat{j})$. Thus

$$\vec{v}_A = \omega_{OA}\hat{k} \times l_{OA}(\cos \theta \hat{i} + \sin \theta \hat{j})$$

We can pull the two scalars out front.

$$\vec{v}_A = \omega_{OA}l_{OA}\hat{k} \times (\cos \theta \hat{i} + \sin \theta \hat{j})$$

Now we use the methodology explained in section ***.*** to perform the cross product.

$$\vec{v}_A = \omega_{OA}l_{OA}(\cos \theta \hat{j} - \sin \theta \hat{i})$$

This implies that $A$ points to the left and up, which is certainly what we would expect. Now we turn our attention to link $AB$. With $\vec{v}_A$ known, we are interested in the velocity of point $B$. Relate the two points.
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\[ \vec{v}_B = \vec{v}_A + \vec{v}_{B/A} \]

\[ \vec{v}_{A/B} = \vec{\omega}_{AB} \times \vec{r}_{A/B} \]

To an observer on \( B \) looking down at \( A \), \( A \) looks as if it is moving to the right. Thus the direction of rotation of link \( AB \) is clockwise—i.e., \( \vec{\omega}_{AB} = \omega_{AB}(\vec{k}) \cdot \vec{r}_{A/B} = l_{AB}(\cos \phi \hat{i} - \sin \phi \hat{j}) \). Thus

\[ \vec{v}_{A/B} = \omega_{AB}(\vec{k}) \times l_{AB}(\cos \phi \hat{i} - \sin \phi \hat{j}) \]

\[ \hat{i} : \quad 0 = -\omega_{OA}l_{OA} \sin \theta + \omega_{AB}l_{AB} \sin \phi \]

\[ \hat{j} : \quad v_B = \omega_{OA}l_{OA} \cos \theta + \omega_{AB}l_{AB} \cos \phi \]

There are two unknowns in these two equations, \( \omega_{ab} \) and \( v_B \), which are the two unknowns sought. Note that the \( \hat{i} \)-equation simply states the fact that the \( \hat{i} \)-component of \( \vec{v}_A \) as calculated from the rotational motion of \( OA \) is the same as the \( \hat{i} \)-component of \( \vec{v}_A \) as calculated from the rotational motion of \( AB \).

**9.5 Rigid-body velocity analysis using velocity diagrams**

A more intuitive, graphical approach is available for this analysis too. It uses velocity diagrams and the fact that no two points on a rigid body can approach or get further away from each other. The body is rigid after all.

Example 9.2 – Slider/Crank mechanism via velocity diagram

At left is shown a stick figure of the slider/crank mechanism. Again, \( \omega_{OA} \) is given as well as the dimensions of the links and the angles \( \theta \) and \( \phi \), as in Example 9.1 The right-hand drawing is of link \( AB \), the link whose motion is unknown. \( \vec{v}_A \) is found as before. With \( \vec{v}_A \) known, it is possible through trigonometry to find \( \omega_{AB} \) and \( v_B \). Here’s how.

Since link \( AB \) is rigid, points \( A \) and \( B \) must have the same component of velocity along the link. We can resolve \( \vec{v}_A \) and \( \vec{v}_B \) into components parallel (\( \parallel \)) and perpendicular (\( \perp \)) to the link.
Figure 9.11 – Slider/Crank velocity diagram

It must be so that

$$\vec{v}_{A\parallel} = \vec{v}_{B\parallel}$$

We can write the trigonometric relationships as illustrated in the figure.

1. $\theta$ is the original angle given for OA
2. $\psi = 90^\circ - \theta$
3. This angle is $\psi$ because it’s the opposite interior angle of 2
4. $\phi$ is the original angle given for AB
5. This is the same $\phi$ between the horizontal and AB
6. $\beta = 90^\circ - \phi$

Thus $\psi$ and $\beta$ can be found. Knowing $v_A$

$$v_{A\parallel} = v_{B\parallel}$$

$$v_A \cos(\phi - \psi) = v_B \cos(\beta)$$

$$v_B = \frac{v_A \cos (\phi - \psi)}{\cos(\beta)}$$

With $v_B$, we can find $v_{B\perp}$. If you were standing on A looking at B, B would look as if it were going to the right. But you would have velocity $v_{A\perp}$ to the left. So the apparent sideways speed of B viewed from A would be $v_{A\perp} + v_{B\perp}$ to the right. Thus

$$\omega_{AB} = \frac{v_{A\perp} + v_{B\perp}}{l_{AB}} \text{ clockwise}$$

This last equation is derived from the well-known relationship for relative velocities

$$\vec{v}_{A/B} = \vec{\omega}_{AB} \times \vec{r}_{A/B}$$

Other approaches are shown in the figure below. The shows two possible velocity diagrams that can be used to find $\vec{v}_B$ and $\vec{v}_{B/A}$ from $\vec{v}_A$. 
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Figure 9.12 – Velocity diagrams for slider/crank mechanism

In the left-hand diagram, \( \vec{v}_A \) is drawn, as well as a line through its tail that is along \( AB \). This allows \( \vec{v}_A \parallel \) and \( \vec{v}_B \) are placed tail to tail. This allows \( \vec{v}_A \parallel \) to be projected onto that line. Of course \( \vec{v}_A \parallel = \vec{v}_B \parallel \). We then draw a vertical line through the tail of \( \vec{v}_A \), since this is the direction of \( \vec{v}_B \). This allows us to draw \( \vec{v}_B \), since \( \vec{v}_B \parallel \) is just its projection. We can then draw a line perpendicular to the direction of \( AB \) through the tail of \( \vec{v}_A \). This allows us to project \( \vec{v}_A \parallel \) and \( \vec{v}_B \) onto this line. With this diagram we can do the trigonometry as follows.

1. Draw \( \theta \) on the diagram
2. \( \psi = 90^\circ - \theta \)
3. \( \phi \) is the original angle given for \( AB \)
4. \( \beta = 90^\circ - \phi \)
5. An inspection of the right angles in the figure shows that this angle is \( \beta \) also

From this, it can be seen that

\[
\vec{v}_{A\parallel} = v_A \cos (\phi - \psi)
\]

\[
v_B = \frac{v_{A\parallel}}{\cos (\beta)}
\]

\[
\vec{v}_{A\perp} = v_A \sin (\phi - \psi), \; \vec{v}_{B\perp} = v_B \sin (\beta)
\]

\[
\omega_{AB} = \frac{v_{A\perp} + v_{B\perp}}{l_{AB}} \text{ clockwise}
\]

In the diagram on the right, a different approach is taken. It is based on the equation

\[
\vec{v}_{B/A} = \vec{v}_B - \vec{v}_A
\]

The only vector known at the outset is \( \vec{v}_A \) – \( -\vec{v}_A \) is drawn. A vertical line is drawn through its head, which represents the direction of \( \vec{v}_B \). The direction of link \( AB \) is drawn, as well as a line perpendicular to this direction. \( \vec{v}_{B/A} \) must lie on this perpendicular line, since point \( A \) doesn’t approach or get farther...
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away from point $B$. With the vertical line and the line perpendicular to $AB$, it is possible to find the vectors $\vec{v}_B$ and $\vec{v}_{B/A}$, after doing a little angle trigonometry. That follows as

1. Draw $\theta$ on the diagram
2. $\psi = 90^\circ - \theta$, since $\theta + 90^\circ + \psi = 180^\circ$
3. $\phi$ is the original angle given for $AB$
4. $\beta = 90^\circ - \phi$
5. An inspection of the right angles in the figure shows that this angle is $\beta$ also

From this diagram

$$v_{B/A} \cos(\beta) = v_A \cos(\psi), \quad v_{B/A} = v_A \frac{\cos(\psi)}{\cos(\beta)}$$

$$\omega_{AB} = \frac{v_{B/A}}{l_{AB}}$$

$$v_B = v_A \sin(\psi) + v_{B/A} \sin(\beta)$$

There are certainly other diagrams, akin to these, that could be drawn to represent the equation relating $\vec{v}_A$ and $\vec{v}_B$. These are just some possibilities.

As can be seen, this approach is more hands-on and graphical. It can be implemented in multiple ways. It involves more trigonometry too. It also requires a different approach with each mechanism, because each mechanism’s geometry is different. Both approaches—the vector approach and the graphical approach—have their advantages and disadvantages. Which to use is up to the dynamicist. Both work. A mixture of the two is also useful.

### 9.6 Instantaneous center

Since a body has just one rotational speed at any point in time, there has to be a point about which the body is rotating at any instant of time. This point is stationary at that point in time, but it can change in the course of time. This seemingly contradictory statement can be understood by referring to the rolling wheel as an example. At any given point in time, the contact point between the wheel and the ground is not moving. It is stationary, but only for that quick instant of time, when it makes contact with the ground. Thus the contact point on the rim of the wheel is the point about which the entire wheel is rotating, but only for an instant. This contact point moves along the ground with the wheel as it rolls. But its velocity is always instantaneously 0.

This is a property of rotation. At any point in time, a body is rotating about an instantaneous center, as described for the wheel above. This is sometimes shortened to instant center or simply i.c. For the wheel the instant center is on the body. But there is no need actually for the instantaneous center to be on the body. Figure 9.3(b) is an example of a body rotating about a point external to it. In 9.3(b), point $O$ is actually a fixed axis of rotation, where the body moves in a circular path about $O$. In this case, the instant center is the permanent center. That is, it does not change in time. A pendulum is another
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example of this. Its pivot point is its permanent center of rotation and, thus, also its instant center, since at every instant, the pendulum is swinging about this pivot point. Many machines have rotating parts whose center of rotation is permanent.

Example 9.3 - Instant center of helicopter rotor

Another interesting example of an instant center is the rotor of a helicopter. The figure below shows a helicopter in normal flight forward at a velocity of $v_O$. On the right the helicopter is shown from above with the rotor athwart the fuselage, perpendicular to the direction of flight. Shown also are the velocities of points along the rotor relative to the rotor axis at $O$. The velocity of a point $C$ is shown, again relative to point $O$. The total velocity at $C$ is

\[
\vec{v}_c = \vec{v}_O + \vec{v}_{C/O}
\]

But as illustrated, $\vec{v}_{C/O} = -\vec{v}_O$, so $\vec{v}_c = 0$. So at this instant, when the rotor is athwart the helicopter fuselage, there is a point to the right of the fuselage along the rotor that is the instant center of the rotor. This point travels along with the helicopter, always at a distance $r_{OC}$ to the right of the helicopter at the speed $v_O$. If the forward speed of the helicopter increases, this point moves outward along the rotor to a point where the aft speed relative to the rotor axis is equal to the forward speed of the helicopter. At hover, when $v_O = 0$, $O$ is the instant (and permanent) center of rotation of the rotor.

Now for the rotor to generate lift, it has to be moving relative to the air. This means that the rotor at point $C$ is generating no lift. Indeed, we can modify the drawing of the helicopter rotor to show the absolute velocity of points along the rotor. These velocities are thus relative to the air through which the helicopter is moving.
Notice that the velocities of points along the rotor to the left of the fuselage are greater than the velocities of points on the right rotor blade. Lift is proportional to the speed of the rotor through air. Because of this, the left rotor blade is generating more lift than the right rotor blade. This makes the helicopter want to roll to the right. Solving this problem was one of the major challenges in the development of helicopter aeronautics.

Example 9.4 - Slider/Crank instant centers

Problems

* Let the rotor of the helicopter described in Example 9.*** be aligned with the longitudinal axis of the helicopter, so that point A is forward of the helicopter and B is aft of point O.

a. What is the instant center of the rotor in this position?

b. How does the instant center of the rotor move—i.e., what path does it take—as the rotor goes through a revolution?

* Where is the instant center of a wheel that is slipping on a car that is still moving forward along an icy surface?