Example 14:

Consider flow between two parallel plates separated by a distance 2H with a uniform heat flux imposed on both plates. The fluid is driven between the plates by an applied pressure gradient in the x-direction. Assume the flow is laminar and fully developed \( (du/dx = 0, v = 0, \text{ and } w = 0) \). (a) Determine the fully developed velocity distribution of the fluid as a function of the mean velocity. (b) Determine the fully developed temperature distribution as a function of the surface and mean temperatures. (c) Determine the Nusselt number for this flow.

**Known:** Pressure driven laminar flow between parallel plates with uniform surface heat flux on both sides, \( H, du/dx = 0, v = 0, \text{ and } w = 0 \)

**Assumptions:**
1. steady flow
2. constant properties
3. Newtonian fluid
4. negligible radiation
5. negligible gravity effects
6. negligible viscous dissipation
7. fully developed
8. negligible end effects
9. conduction in y-direction much greater than conduction in x-direction

**Find:** (a) \( u(y) \) and \( u_m \), (b) \( T(y) \) and \( T_m \), (c) \( Nu \)

**Solution:**

**Answer:**

(a) \( u = \frac{3}{2} u_m \left[ 1 - \left( \frac{y}{H} \right)^2 \right] \), \( u_m = -\frac{H^2}{3 \mu} \frac{dp}{dx} \)

(b) \( T = T_s - \frac{35}{136} (T_s - T_m) \left[ 5 - 6 \left( \frac{y}{H} \right)^2 + \left( \frac{y}{H} \right)^4 \right], \quad (T_s - T_m) = \frac{17}{35} \frac{u_m H^2}{\alpha} \frac{dT_m}{dx} \)

(c) \( Nu = 8.23 \)
(a) Derivation of velocity distribution, \(u(y)\), and mean velocity, \(u_m\)

**Conservation of mass** for constant property flow:

\[
\nabla \cdot \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0
\]

For fully developed flow, \(v = w = 0\), thus

\[
\frac{\partial u}{\partial x} = 0 \quad \text{and} \quad u(y)
\]

**Momentum equation** for constant property flow of a Newtonian fluid:

\[
\rho \left( \frac{\partial \vec{V}}{\partial t} + \vec{V} \cdot \nabla \vec{V} \right) = \mu \nabla^2 \vec{V} - \nabla p + \rho \vec{g}
\]

\(x\)-component for 2-D steady flow in Cartesian coordinates:

\[
\rho \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - \frac{\partial p}{\partial x} + F_x
\]

\[
\mu \frac{d^2 u}{dy^2} = \frac{\partial p}{\partial x}
\]

\(y\)-momentum reduces to \(\partial p/\partial y = 0\), so \(p(x)\) and

\[
\frac{d^2 u}{dy^2} = \frac{1}{\mu} \frac{dp}{dx}
\]

Integrate twice to get:

\[
u(y) = \frac{1}{\mu} \frac{dp}{dx} \frac{y^2}{2} + C_1 y + C_2
\]

Impose boundary conditions: \(u(-H) = 0\) and \(u(H) = 0\) (no-slip at wall):

\[
u(-H) = \frac{H^2}{2\mu} \frac{dp}{dx} - H C_1 + C_2 = 0
\]

\[
u(H) = \frac{H^2}{2\mu} \frac{dp}{dx} + H C_1 + C_2 = 0
\]
Adding these equations we get:

\[ C_2 = -\frac{H^2}{2\mu} \frac{dp}{dx} \quad \text{and} \quad C_1 = 0 \]

\[ u(y) = -\frac{H^2}{2\mu} \frac{dp}{dx} \left[ 1 - \left( \frac{y}{H} \right)^2 \right] \]

Thus, the velocity profile is parabolic.

Recall definition for mean velocity, \( u_m \), and mass flow rate, \( \dot{m} \):

\[ \dot{m} = \rho \ u_m \ A = \int_A \rho \ u \ dA \]

\[ u_m = \frac{1}{\rho A} \int_A \rho \ u \ dA \]

Returning to flat plate example where \( W \) is the width of the plate perpendicular to flow:

\[ u_m = \frac{1}{\rho (2H) W} \int_0^W \int_{-H}^H \rho \ u \ dy \ dz = \frac{1}{H} \int_0^H \frac{H^2}{2\mu} \frac{dp}{dx} \left[ \left( \frac{y}{H} \right)^2 - 1 \right] dy \]

\[ u_m = H \frac{dp}{2\mu} \left( \frac{y^3}{3H^2} - y \right) \bigg|_0^H = H^2 \frac{dp}{2\mu} \left( \frac{1}{3} - 1 \right) \]

\[ u_m = -\frac{H^2}{3\mu} \frac{dp}{dx} \]

\[ \frac{dp}{dx} = -\frac{3\mu}{H^2 u_m} \]

Substituting into velocity profile:

\[ u(y) = \frac{3}{2} u_m \left[ 1 - \left( \frac{y}{H} \right)^2 \right] \]
(b) Derivation of temperature distribution, \( T(x, y) \), and mean temperature, \( T_m \)

**Energy equation** for constant property flow of a Newtonian fluid:

\[
\rho \ c_p \left( \frac{\partial T}{\partial t} + \vec{V} \cdot \nabla T \right) = k \ \nabla^2 T + \mu \ \Phi + \dot{q}
\]

For 2-D steady flow in Cartesian coordinates:

\[
\rho \ c_p \left( u \ \frac{\partial T}{\partial x} + v \ \frac{\partial T}{\partial y} \right) = k \ \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) + \mu \ \Phi + \dot{q}
\]

\[
\frac{u}{\alpha} \ \frac{\partial T}{\partial x} = \frac{\partial^2 T}{\partial y^2} \quad \text{where we assume} \quad \frac{\partial^2 T}{\partial y^2} \gg \frac{\partial^2 T}{\partial x^2}
\]

Substitute in \( u(y) = \frac{3}{2} \ u_m \left[ 1 - \left( \frac{y}{H} \right)^2 \right] \):

\[
\frac{\partial^2 T}{\partial y^2} = \frac{3}{2} \ \frac{u_m}{\alpha} \ \frac{dT_m}{dx} \left[ 1 - \left( \frac{y}{H} \right)^2 \right]
\]

For constant surface heat flux recall that: \( \frac{\partial T}{\partial x} = \frac{dT_m}{dx} = \text{constant} \)

Integrate twice to obtain:

\[
T = \frac{3}{2} \ \frac{u_m}{\alpha} \ \frac{dT_m}{dx} \left( \frac{y^2}{2} - \frac{y^4}{12 \ H^2} \right) + C_1 \ y + C_2
\]

Impose boundary conditions \( T(-H) = T(+H) = T_s \) (note that \( T_s \) is unknown):

\[
T(H) = \frac{3}{2} \ \frac{u_m}{\alpha} \ \frac{dT_m}{dx} \left( \frac{H^2}{2} - \frac{H^2}{12} \right) + C_1 \ H + C_2 = T_s
\]

\[
T(-H) = \frac{3}{2} \ \frac{u_m}{\alpha} \ \frac{dT_m}{dx} \left( \frac{H^2}{2} - \frac{H^2}{12} \right) - C_1 \ H + C_2 = T_s
\]

\[
\Rightarrow C_2 = T_s - \frac{3}{2} \ \frac{u_m}{\alpha} \ \frac{H^2}{12} \ \frac{dT_m}{dx} \left( \frac{5}{12} \right) \quad \text{and} \quad \Rightarrow C_1 = 0
\]
\[ T_s - T = \frac{3}{2} \frac{u_m H^2}{\alpha} \frac{dT_m}{dx} \left[ \frac{5}{12} - \frac{1}{2} \left( \frac{y}{H} \right)^2 + \frac{1}{12} \left( \frac{y}{H} \right)^4 \right] \]

Recall definition for mean temperature:

\[ T_m = \frac{1}{\rho u_m A c_v} \int_A \rho u c_v T \, dA = \frac{1}{2 u_m} \int_0^H u \, T \, dy \]

\[ T_m = \frac{1}{u_m H} \int_0^H \left\{ \frac{3}{2} u_m \left[ 1 - \left( \frac{y}{H} \right)^2 \right] \right\} \left\{ T_s - \frac{3}{2} \frac{u_m H^2}{\alpha} \frac{dT_m}{dx} \left[ \frac{1}{12} \left( \frac{y}{H} \right)^4 - \frac{1}{2} \left( \frac{y}{H} \right)^2 + \frac{5}{12} \right] \right\} dy \]

After many steps:

\[ (T_s - T_m) = \frac{17}{35} \frac{u_m H^2}{\alpha} \frac{dT_m}{dx} \]

\[ \frac{(T_s - T)}{(T_s - T_m)} = \frac{35}{136} \left[ 5 - 6 \left( \frac{y}{H} \right)^2 + \left( \frac{y}{H} \right)^4 \right] \]

(c) Derivation of Nusselt number, \( \text{Nu} \)

Recall definition of heat transfer coefficient:

\[ h = \frac{k_f}{(T_s - T_m)} \left. \frac{\partial T}{\partial y} \right|_{y=H} \]

\[ \text{Nu} = \frac{h D_h}{k_f} = \frac{D_h}{(T_s - T_m)} \left. \frac{\partial T}{\partial y} \right|_{y=H} \quad \text{where} \quad D_h = \frac{4 A_s}{P} = \frac{4 (2H) W}{2W} = 4H \]

\[ \text{Nu} = \frac{4H}{(T_s - T_m)} \left. \frac{\partial}{\partial y} \left[ T_s - \frac{35}{136} (T_s - T_m) \left[ \left( \frac{y}{H} \right)^4 - 6 \left( \frac{y}{H} \right)^2 + 5 \right] \right] \right|_{y=H} \]

\[ \text{Nu} = \frac{4H}{(T_s - T_m)} \left. \left[ -\frac{35}{136} (T_s - T_m) \left[ 4 \left( \frac{y}{H} \right)^3 - 12 \left( \frac{y}{H} \right)^2 \right] \right] \right|_{y=H} \]

\[ \text{Nu} = 4H \left( -\frac{35}{136} \frac{(-8)}{H} \right) = \frac{140}{17} = 8.2353 \quad \text{(Same as in Table 8.1)} \]