Reynolds-Averaged Navier-Stokes (RANS) equations

Navier Stokes equations (for Cartesian coordinates in conservative form with no body forces and assuming an incompressible fluid):

\[ \rho \left[ \frac{\partial u}{\partial t} + \frac{\partial (u^2)}{\partial x} + \frac{\partial (uv)}{\partial y} + \frac{\partial (uw)}{\partial z} \right] = \frac{\partial}{\partial x} \left( \mu \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left( \mu \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial z} \left( \mu \frac{\partial u}{\partial z} \right) - \frac{\partial p}{\partial x} \]

\[ \rho \left[ \frac{\partial v}{\partial t} + \frac{\partial (uv)}{\partial x} + \frac{\partial (v^2)}{\partial y} + \frac{\partial (vw)}{\partial z} \right] = \frac{\partial}{\partial x} \left( \mu \frac{\partial v}{\partial x} \right) + \frac{\partial}{\partial y} \left( \mu \frac{\partial v}{\partial y} \right) + \frac{\partial}{\partial z} \left( \mu \frac{\partial v}{\partial z} \right) - \frac{\partial p}{\partial y} \]

\[ \rho \left[ \frac{\partial w}{\partial t} + \frac{\partial (uw)}{\partial x} + \frac{\partial (vw)}{\partial y} + \frac{\partial (w^2)}{\partial z} \right] = \frac{\partial}{\partial x} \left( \mu \frac{\partial w}{\partial x} \right) + \frac{\partial}{\partial y} \left( \mu \frac{\partial w}{\partial y} \right) + \frac{\partial}{\partial z} \left( \mu \frac{\partial w}{\partial z} \right) - \frac{\partial p}{\partial z} \]

Write using Einstein notation (sum each repeated index over \(i, j,\) and \(k\)) for Cartesian coordinates and the \(x\)-component where \((x, x, x) = (x, y, z)\) and \((u, u, u) = (u, v, w)\):

\[ \rho \left[ \frac{\partial u_i}{\partial t} + \frac{\partial (u_i u_j)}{\partial x_j} \right] = \frac{\partial}{\partial x_j} \left( \mu \frac{\partial u_i}{\partial x_j} \right) - \frac{\partial p}{\partial x_i} \]

Alternatively, in terms of mean strain rate, \(s_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)\), where for incompressible flow by conservation of mass \(\frac{\partial u_j}{\partial x_j} = 0\):

\[ \rho \left[ \frac{\partial u_i}{\partial t} + \frac{\partial (u_j u_i)}{\partial x_j} \right] = \frac{\partial}{\partial x_j} \left( 2 \mu s_{ij} \right) - \frac{\partial p}{\partial x_i} \]

Define mean and fluctuating components for velocity and pressure:

\[ u_i(x, j, x_k, t) = \overline{u}_i(x, j, x_k) + u'_i(x, j, x_k, t), \quad p(x, j, x_k, t) = \overline{p}(x, j, x_k) + p'(x, j, x_k, t) \]

Substitute in mean and fluctuating components and expand to get:

\[ \rho \left[ \frac{\partial (\overline{u}_i + u'_i)}{\partial t} + \frac{\partial [(\overline{u}_j + u'_j)(\overline{u}_i + u'_i)]}{\partial x_j} \right] = \frac{\partial}{\partial x_j} \left[ \mu \frac{\partial (\overline{u}_i + u'_i)}{\partial x_j} \right] - \frac{\partial (\overline{p} + p')}{\partial x_i} \]

\[ \rho \left[ \frac{\partial \overline{u}_i}{\partial t} + \frac{\partial (u'_i)}{\partial t} + \frac{\partial \overline{u}_j}{\partial x_j} + \frac{\partial (\overline{u}_j u'_i)}{\partial x_j} + \frac{\partial (\overline{u}_j u'_i)}{\partial x_j} + \frac{\partial (u'_j u'_i)}{\partial x_j} \right] = \frac{\partial}{\partial x_j} \left[ \mu \left( \frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial u'_i}{\partial x_j} \right) - \frac{\partial \overline{p}}{\partial x_i} - \frac{\partial p'}{\partial x_i} \right] \]
Time average equations and apply the following rules:

\[ \bar{u}_i = u_i \]
\[ \bar{u}_i + u'_i = \bar{u}_i + u'_i = \bar{u}_i \]
\[ \bar{u}_i \cdot u'_j = \bar{u}_i \cdot u'_j = 0 \]
\[ \frac{\partial \bar{u}_i}{\partial x_j} = \frac{\partial \bar{u}_i}{\partial x_j} \]
\[ \bar{u}_i^2 = \bar{u}_i^2 \]
\[ u'_i u'_j < 0 \]

Many terms cancel to give Reynolds-averaged Navier-Stokes (RANS) equations:

\[ \rho \left[ \frac{\partial \bar{u}_i}{\partial t} + \frac{\partial (\bar{u}_j \bar{u}_i)}{\partial x_j} \right] = \frac{\partial}{\partial x_j} \left[ \mu \left\{ \frac{\partial \bar{u}_i}{\partial x_j} \right\} - \rho \bar{u}_j u'_i \right] - \frac{\partial \bar{p}}{\partial x_i} \]

\[ \rho \left[ \frac{\partial \bar{u}_i}{\partial t} + \frac{\partial (\bar{u}_j \bar{u}_i)}{\partial x_j} \right] = \frac{\partial}{\partial x_j} \left[ 2 \mu \bar{x}_j - \rho \bar{u}_j u'_i \right] - \frac{\partial \bar{p}}{\partial x_i} \]

**NOTE:** RANS equations have additional shear stress terms due to turbulent mixing.

*Momentum transfer* is by two mechanisms:
1. Viscosity (or friction) between moving adjacent fluid layers; *microscopic* effect (due to rubbing of adjacent molecules) referred to as *diffusion of momentum*
2. Bulk fluid motion between different flow regions due to average and fluctuating velocity components; *macroscopic* mixing effect referred to as *advection of momentum*; fluctuating component results in apparent stress called *Reynolds Stresses* (after Osburn Reynolds in 1880’s)

\[ \tau'_i = \tau'_j = -\rho \bar{u}_i \bar{u}_j \quad \text{(where } i = j \text{ for normal stress)} \]

**NOTE:** Reynolds stresses are positive because the velocity fluctuations are correlated through conservation of mass such that \( \bar{u}_i u'_j < 0 \).

*Boussinesq Hypothesis* defines turbulent viscosity, \( \mu_t \), or eddy viscosity, \( \epsilon_M = \mu_t / \rho \):

\[ \tau'_i = -\rho \bar{u}_i \bar{u}_j = \mu_t \left( \frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right) - \frac{2}{3} \left( \rho k + \mu_t \frac{\partial \bar{u}_i}{\partial x_j} \right) \delta_{ij}, \quad \delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \]

\[ \frac{\partial \bar{u}_i}{\partial t} + \frac{\partial (\bar{u}_j \bar{u}_i)}{\partial x_j} = \frac{\partial}{\partial x_j} \left[ (\nu + \epsilon_M) \left( \frac{\partial \bar{u}_i}{\partial x_j} \right) \right] - \frac{1}{\rho} \frac{\partial \bar{p}}{\partial x_i} \]

\[ \frac{\partial \bar{u}_i}{\partial t} + \frac{\partial (\bar{u}_j \bar{u}_i)}{\partial x_j} = \frac{\partial}{\partial x_j} \left[ 2 (\nu + \epsilon_M) \bar{x}_j \right] - \frac{1}{\rho} \frac{\partial \bar{p}}{\partial x_i} \]

\[ \bar{x}_j = \frac{1}{n} \sum \bar{u}_j \]