

UNIVERSITY of CALIFORNIA
SANTA CRUZ

**M-THEORY, SUPERGRAVITY AND THE MATRIX MODEL:
GRAVITON SCATTERING AND NON-RENORMALIZATION
THEOREMS**

A dissertation submitted in partial satisfaction of the
requirements for the degree of

DOCTOR OF PHILOSOPHY

in

PHYSICS

by

Robert S. Echols

June 1999

The dissertation of Robert S. Echols is approved:

Professor Michael Dine, Chair

Professor Howard Haber

Professor Abraham Seiden

Dean of Graduate Studies

Copyright © by

Robert S. Echols

1999

Contents

Abstract	v
Acknowledgements	vi
List of Figures	vii
1 Introduction	1
1.1 Motivation	1
1.2 From String Theory to M-Theory	4
1.3 Matrix Model Description of M-Theory	6
2 Evidence for the Matrix Conjecture: Graviton-Graviton Scattering	11
2.1 Matrix Model low energy effective action for $SU(2)$	12
2.1.1 Non-Renormalization of F_{0i}^2	15
2.1.2 Coefficient of v^4/r^7	16
2.2 Supergravity Effective Action: Two Graviton Interaction	18
2.3 Importance of Non-Renormalization Theorems	21
3 Evidence of a Non-Renormalization Theorem for v^4 terms	23
3.1 Infrared Divergences in $SU(2)$	27
3.2 Finite Renormalizations in $SU(3)$	32
4 Multigraviton Scattering	35
4.1 Background	37
4.2 Matrix Model low energy effective action for $SU(3)$	39
4.2.1 Terms with no derivatives	43
4.2.2 Cancellation of $(F_{0i}^a)^2$ or $(\dot{X}_i^a)^2$	45
4.2.3 V^4/R^7	45
4.2.4 $V^4 x^2/R^9$ and $V^4 v^2/R^9 r^5$	46
4.3 Comment on the Eikonal approximation	47
4.4 Computing the S -Matrix in the Matrix Model	49
4.4.1 Path integral Computation of the S -Matrix	52
4.4.2 The Ladder Graphs	54
4.5 Additional contributions to three graviton scattering	56

4.6	More Gravitons	59
4.6.1	n-Graviton Scattering	59
4.6.2	Other Dimensions	63
4.7	Some Puzzles	64
5	Non-Renormalization Theorems for v^{2N} terms in $SU(N)$	67
5.1	Non-renormalization of eight fermion terms in $SU(2)$	68
5.2	Supersymmetric completion of $\frac{v_3^4}{r^6}(\vec{v}_{12} \cdot \nabla_R)^2 \frac{1}{R^7}$	70
5.3	Twelve fermion term with the fewest factors of R ?	72
5.4	Outline of a Direct Proof for v^{2N} terms in $SU(N)$	74
6	Conclusion	77
	Bibliography	80

Abstract

M-Theory, Supergravity and the Matrix Model: Graviton Scattering and Non-Renormalization Theorems

by

Robert S. Echols

After briefly reviewing M-Theory and its relationship to the previously known superstring theories, we investigate some of the initial evidence for the matrix model description of M-Theory, namely graviton-graviton scattering. We discuss the importance of non-renormalization theorems in understanding the matrix models success for reproducing the supergravity two-graviton scattering result for finite N , and provide evidence for the non-renormalization theorem involving terms with four derivatives in the low energy matrix model effective action.

We then go on to analyze the matrix models ability to reproduce tree level supergravity amplitudes for multigraviton scattering. Beginning with three-graviton scattering, we discuss and resolve the apparent discrepancy between the supergravity and matrix model amplitudes. We also show exact agreement for certain terms in n -graviton scattering. The matrix model's success in describing these terms in n -graviton scattering led us to search for an infinite sequence of non-renormalization theorems for a subset of terms in the matrix model effective action. We describe the difficulties with some of our approaches in achieving this goal.

Acknowledgements

I am extremely grateful to have had the opportunity to work with my advisor, Michael Dine. His breath and depth of knowledge is illuminating and very educational. Michael's ability to enthusiastically get to the heart of a problem with speed and clarity has been inspirational.

I would like to thank Josh Gray for the numerous physics discussions we have shared. He has made learning the details of M-theory and its matrix description much more fun and interesting.

I am grateful to Abe Seiden for all the material I learned in my first particle physics course. I would like to thank Howie Haber for his masterful lectures in our weak interaction course. I would also like to extend my thanks to all the other professors who have shared their knowledge with me.

I would like to thank David Belanger for obtaining GAANN funding and the graduate committee for offering me a GAANN fellowship and a Regents scholarship.

Finally I want to express appreciation to my immediate family, Michele and Asali, for supporting my desire to learn more physics.

List of Figures

2.1	Graviton-graviton scattering at small momentum transfer. Solid lines are the scattering gravitons. The wiggly line represents a virtual graviton with zero longitudinal momentum.	19
3.1	Infrared divergent contributions to the effective action.	25
3.2	Some two loop corrections to the effective action.	29
3.3	Three loop correction to the effective action.	32
4.1	Ladder contribution to the supergravity amplitude. Solid lines are the scattering gravitons. Wiggly lines represent virtual gravitons with zero longitudinal momentum.	50
4.2	Corresponding contribution in the matrix model involving iteration of the one loop Hamiltonian. Dashed lines represent matrix elements of the interaction potential.	51
4.3	Matrix model contribution to three graviton amplitude.	56
4.4	Contribution to three graviton amplitude involving three graviton vertex. .	58
4.5	A matrix model diagram contributing to four graviton scattering.	60
4.6	Another diagram which is easily computed.	61
4.7	A supergravity contribution to four graviton scattering.	62
4.8	Diagram contributing to n graviton scattering.	63

Chapter 1

Introduction

1.1 Motivation

Perhaps the most naive motivation for string theory is reflected in the fact that present day particle physicists treat the fundamental particles as point objects in their calculations. It seems natural to suppose that the fundamental objects have a finite size. String theory does exactly this by postulating that the building blocks of nature are tiny pieces of string. However, with the relatively recent discovery of D-branes [1, 2] as essential fundamental objects in string theory along with the emergence of M-theory [3, 4] and its fundamental two and five dimensional membranes, it is clear that strings are not necessarily the fundamental objects either.

Aside from the conceptual problems of treating the fundamental particles as point-like, there are technical difficulties as well. The most notorious problem is the appearance of ultra-violet or high energy divergences in four dimensional perturbative quantum field theory calculations at one loop. Although these difficulties can be overcome for the quantum

field theories describing the strong, weak and electromagnetic forces, the quantization of Einstein's theory of general relativity leads to uncontrollable divergences, signaling the need for an underlying fundamental theory. String theory is able to soften all the ultraviolet divergences and at present is the only known consistent theory of quantum gravity. Superstring theory is more than a theory of quantum gravity, it also has the ability to unify all the fundamental forces found in nature [5, 6].

An important ingredient in superstring theory is the need for nature to exhibit spacetime supersymmetry, a symmetry relating the equal numbers of bosonic and fermionic degrees of freedom in the theory [7]. Low energy supersymmetry is currently the most popular means for stabilizing the hierarchy from the electroweak scale (100 GeV) to the GUT scale (10^{16} GeV) with the added benefit of nicely predicting the unification of the gauge coupling constants. While supersymmetry is an essential part of superstring theory, low energy supersymmetry breaking is at present not yet a prediction of the theory and supersymmetry broken at high energies is also a consistent possibility.

Taking the approach that superstring theory is possibly the underlying consistent theory which will replace the low energy effective theory of the standard model while incorporating quantum gravity, we investigate in this thesis the matrix model description of M-theory, a recently discovered non-perturbative limit of superstring theory. By way of introduction, the next section of this chapter describes the emergence of eleven dimensional M-theory as the strong coupling limit of the Type IIA superstring. We finish this chapter by giving a heuristic motivation for the matrix model description of M-theory and discuss some interesting features of the matrix model.

In chapter two we review the calculations which show how matrix theory does

indeed reproduce the leading term for long distance graviton-graviton scattering in supergravity. The matrix theory calculations are done using the background field method with an unspecified background allowing us to determine if acceleration terms are present in the matrix model effective action with four external background fields. The supergravity calculation is carried out by using the Feynman rules derived from the Einstein-Hilbert action in eleven dimensions. We conclude this chapter with a closer look at the $SU(2)$ matrix model loop expansion and discuss the role non-renormalization theorems play in understanding the matrix model's ability to reproduce the two graviton scattering amplitude.

Chapter three analyzes the matrix model effective action for $SU(2)$ and $SU(3)$ beyond one loop. By investigating potential infrared divergences, we show that divergent contributions to v^4 terms at two loops cancel. In fact, we are able to show to all orders that the infrared corrections to v^4 cancel as a result of $\frac{1}{r^7}$ being the Green's function for the nine dimensional Laplacian, further establishing evidence for the non-renormalization of v^4 terms in the matrix model effective action. We then generalize to $SU(3)$ and show that potentially finite corrections to the v^4 terms cancel at two loops. Unlike the $SU(2)$ case we are unable to establish an all orders argument.

In chapter four we take a more in depth look at multigraviton scattering. Providing a rebuttal to a claim in the literature, we describe in detail why the matrix model effective action does not contain a term found in the supergravity scattering amplitude for three gravitons. We then go on to show how the matrix model can generate such a term by analyzing the matrix model S -matrix. We look at additional terms in three graviton scattering and then generalize to n -graviton scattering, showing agreement between the matrix model and supergravity with up to four spacetime dimensions compactified. At the end of

this chapter we discuss renormalized terms which are subleading in the velocity expansion. In particular, we look at a renormalized v^6 term in $SU(4)$ and conclude that there are many similar terms in the v^{2N} velocity expansion with fewer powers of velocity in $SU(N)$ starting at three loops.

Based on our success with using the matrix model to describe certain terms in n -graviton scattering we explore in chapter five various attempts at finding an infinite sequence of non-renormalization theorems. We begin by reviewing the techniques used to prove a non-renormalization theorem for the v^4 term in $SU(2)$. In generalizing to $SU(3)$, we encounter difficulties with a proliferation of fermion tensor structures and the need to assume that certain fermion terms involving acceleration are absent in the matrix model effective action. We are, however, able to provide a direct proof showing a certain class of v^{2N} terms in the $SU(N)$ matrix model are exact by making some reasonable assumptions.

1.2 From String Theory to M-Theory

What is superstring theory? In the mid 1980's, the answer was five consistent string theories each formulated in ten space-time dimensions possessing world-sheet and space-time supersymmetry with a well defined perturbation expansion. The five theories have become known as the type IIA, type IIB, type I, $E_8 \times E_8$ heterotic and the $SO(32)$ heterotic. The type II and heterotic theories describe oriented closed strings while the type I theory describes open and closed unoriented strings¹. The size of these strings is taken to be close to the Planck scale (10^{-33} cm) and the six extra dimensions are believed to have remained small while our familiar three spatial dimensions have expanded since the time of

¹An unoriented string does not have an orientation or handedness. Speaking more precisely, if the string has length, l , and is parameterized by σ , then $\sigma = l - \sigma$ is a symmetry of an unoriented string.

the big bang. In this scenario, the fundamental particles observed in nature are excitations of the quantized string. For example, the graviton turns out to be the first excitation of the left and right moving modes of a closed string.

Recently, in what has been called the second superstring revolution², new methods have been developed to understand superstring theories at strong coupling. What has emerged is an understanding that the five distinct superstring theories are merely weak coupling limits of a larger non-perturbative eleven dimensional limit called, M-theory³. Crucial to these developments has been the discovery of new dynamical objects in string theory, *Dp*-branes [1], *p*-dimensional objects on which open strings with Dirichlet ($X^\mu = \text{constant}$) boundary conditions can end. The first (and most relevant for the discussion below) route to M-theory from string theory was discovered by Witten [3], as the strong coupling limit of the IIA superstring. It had previously been known that the low energy limit of the IIA string, type IIA supergravity, could be obtained by the dimensional reduction of the elegantly formulated $\mathcal{N} = 1$ supergravity in eleven dimensions [9]. The spectrum of eleven dimensional supergravity contains 128 bosonic degrees of freedom in the form of the graviton and an antisymmetric 3-Form tensor, and 128 fermionic gravitino degrees of freedom. The natural question to ask is what consistent theory could eleven dimensional supergravity be the low energy limit of? In analyzing the *D0*-branes of the IIA string theory, one finds that the mass of *N* of them behaves as

$$M = \frac{N}{gl_s} \tag{1.1}$$

where *g* is the coupling and *l_s* is the string length. Since *D0*-branes are annihilated by half

²See, for example, the article by J. Schwarz [8].

³Some authors argue that M-theory is the underlying theory from which all the rest of string theory is to be derived while others contend that it is merely another point in the moduli space of the theory.

the supersymmetry generators (BPS states), the mass relation above is exact for any value of the coupling. Taking the $g \rightarrow \infty$ limit the spectrum goes over to a continuum reminiscent of Kaluza-Klein modes when a dimension is uncompactified. Making the identification

$$R_{11} = gl_s, \tag{1.2}$$

we see that a new eleventh dimension appears in the strong coupling limit of the IIA string theory. We can also see why this extra dimension and the $D0$ -branes went unnoticed in weak coupling string theory. In the limit $g \rightarrow 0$ the $D0$ -branes become infinitely massive and the eleventh dimension goes to zero size. The identification (1.2) can be made stronger by realizing that these states (1.1) are the 256 states of eleven dimensional supergravity compactified on radius R_{11} . Thus, the IIA theory at strong coupling grows another dimension and this new limit, called M-theory, contains eleven dimensional supergravity at low energy.

1.3 Matrix Model Description of M-Theory

To arrive at the original conjecture of M-theory as a matrix model [10], Banks, Fischler, Shenker, and Susskind (BFSS) exploited the M-theory/IIA duality mentioned in the previous section. In particular, they argued that a spatial compactification of an eleven dimensional coordinate in M-theory on a circle of radius, R_{11} , gives rise to a quantized momentum, $P_{11} = N/R_{11}$. In the infinite momentum frame (IMF), $N \rightarrow \infty$ limit, they assumed that objects of negative and zero momentum decouple, leaving only objects which carry positive momentum. Since M-theory compactified on a circle is the IIA string, and only $D0$ -branes carry P_{11} in the IIA theory, BFSS concluded that M-theory in the limit

$N \rightarrow \infty$ must be described by the theory of N $D0$ -branes. Previous work by Witten [11] had established that the theory of N $D0$ -branes is described by a $U(N)$ supersymmetric quantum mechanics derived from the dimensional reduction of $9 + 1$ dimensional super Yang-Mills theory down to $0 + 1$ dimensions. The BFSS conjecture can be summarized as this: M-theory in the IMF is a $U(N)$ supersymmetric quantum mechanics (matrix model) describing $D0$ -branes in the limit $N \rightarrow \infty$.

In addition to their heuristic motivation for the matrix model description of M-theory, BFSS provided a number of pieces of evidence to support the conjecture. They showed that the matrix model contained the 256 states of the supergravity multiplet in addition to being able to describe large classical membranes both of which were believed to exist in M-theory. BFSS also presented a calculation showing that graviton scattering in the matrix model at low energy and long distance gives what one expects of M-theory in this regime, namely graviton scattering in supergravity. To date, numerous additional pieces of evidence have been put forth to support the BFSS conjecture, many of which can be found in the reviews of the matrix model [12, 13, 14, 15].

Shortly after the original conjecture of BFSS, another conjecture was put forward by Susskind [16] arguing for an equivalence between M-theory and the matrix model for finite N . Susskind noted that if a light-like coordinate $x^- = x_0 - x_{11}$ is compactified then the states with negative discrete momentum, $p_- = \frac{N}{R}$ decouple for all N . Periodically identifying a light-like coordinate is known in the literature as Discrete Light Cone Quantization (DLCQ) [17]. Consequently, Susskind's conjecture states that the DLCQ of M-theory is described by the $U(N)$ super Yang-Mills matrix theory for finite N . This new conjecture was subsequently derived by considering M-theory compactified on a light-like circle as a limit of a small

spatially compactified circle boosted by a large amount [18, 19]. However, this does not necessarily mean that for long distance processes, the finite N matrix model should agree with the the DLCQ of eleven dimensional supergravity. In other words, is the DLCQ of M-theory described at low energy by DLCQ supergravity? We will refer to this expectation as the “naive DLCQ”. Throughout the course of this thesis our primary emphasis will be placed on testing the naive DLCQ hypothesis. We will investigate the matrix model’s ability to reproduce graviton scattering and the role of non-renormalization theorems in understanding the agreement with supergravity. Before beginning it will be useful to discuss the matrix model description of $D0$ -branes.

Prior to the BFSS proposal, $D0$ -brane dynamics had been studied by a number of authors [20, 21, 22]. As mentioned above, the supersymmetric quantum mechanics describing $D0$ -branes is obtained by the dimensional reduction of ten dimensional super Yang-Mills theory down to one time dimension giving the action,

$$S = \int dt \left[\frac{1}{g} \text{tr}(D_t X^i D_t X^i) + \frac{1}{2g} M^6 R_{11}^2 \text{tr}([X^i, X^j][X^i, X^j]) + \right. \quad (1.3)$$

$$\left. \frac{1}{g} \text{tr}(i\theta^T D_t \theta + M^3 R_{11} \theta^T \gamma^i [X^i, \theta]) \right]$$

where R_{11} is the eleven dimensional radius, M is the eleven dimensional Planck mass and $g = 2R_{11}$. The diagonal elements of the hermitian matrices X^i give the positions of the $D0$ -branes in the transverse space ($i = 1$ to 9). There are sixteen real fermionic coordinates, θ_a , and γ_{ab}^i are the 16×16 real symmetric Dirac matrices representing the $SO(9)$ Clifford algebra. The covariant derivative is defined by $D_t X^i = \partial_t X^i + [A, X^i]$, where A is the gauge field and a similar expression holds for $D_t \theta$.

To see how the matrix model reproduces the supergravity multiplet of 256 states,

consider the free matrix model Hamiltonian (choosing the gauge $A = 0$),

$$H = \frac{1}{g} \text{tr}(\partial_t X^i \partial_t X^i). \quad (1.4)$$

Since (1.4) is independent of fermions, we can form a Clifford algebra with the sixteen degenerate fermions

$$\{\theta_\alpha, \theta_\beta\} = \delta_{\alpha,\beta}. \quad (1.5)$$

Making eight ($i = 1$ to 8) creation and annihilation operators from the fermions

$$a_i^\pm = \frac{1}{\sqrt{2}}(\theta_{2i-1} \pm \theta_{2i}), \quad (1.6)$$

we can act on the ground state using the usual Fermi-Dirac statistics to show there are a total of 256 states. With further analysis [23] it can be shown that these 256 states decompose into a **44+84+128**, the spin content of eleven dimensional supergravity.

An interesting feature of the matrix model is the potential term involving the coordinates,

$$\text{tr}([X^i, X^j][X^i, X^j]). \quad (1.7)$$

At long distances, large values of X , this potential must vanish classically to minimize the energy of the system. The X 's must be diagonal and commuting of the form

$$\vec{x} = \begin{pmatrix} \vec{x}_1 & 0 & \dots & 0 \\ 0 & \vec{x}_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \vec{x}_N \end{pmatrix}. \quad (1.8)$$

Commuting coordinates are, of course, what we are used to, but when the $D0$ -branes get close to each other the dynamics is taken over by the full non-commuting geometry. For our purposes of long distance scattering, we will be interested in classically commuting

coordinates with small non-commuting quantum fluctuations. The simplest case to analyze, which we will do in greater depth in the next chapter, is two $D0$ -branes separated by a distance r . After removing an overall $U(1)$ describing the center of mass motion, we are left with an $SU(2)$ theory with classical positions of the $D0$ -branes given by

$$\vec{x} = r T_3 = \frac{1}{2} \begin{pmatrix} \vec{r} & 0 \\ 0 & -\vec{r} \end{pmatrix} \quad (1.9)$$

and the relative separation between the $D0$ -branes is $\vec{x}_1 - \vec{x}_2 = \vec{r}$. The classical $D0$ -brane separation acts as a vacuum expectation value (vev) and breaks $SU(2)$ down to $U(1)$ in the same way the Higgs vev provides symmetry breaking in the standard model. The off-diagonal states become very massive for large r and can be integrated out giving rise to an effective potential for the massless diagonal degrees of freedom describing the $D0$ -brane positions. We will see in the next chapter, that the leading term in the effective potential has the necessary form and precise numerical coefficient to correspond with graviton-graviton scattering in eleven dimensional supergravity.

Chapter 2

Evidence for the Matrix

Conjecture: Graviton-Graviton

Scattering

One of the important original pieces of evidence for the matrix model conjecture was that it successfully reproduced graviton-graviton scattering in supergravity [10]. It will be useful to review in detail the calculations showing agreement between the matrix model and supergravity for graviton-graviton scattering. The technique we employ to calculate the leading term in the matrix model effective action allows us to determine if acceleration terms are present. Other matrix model calculations appearing in the literature either lacked detail [24] to determine the presence of acceleration terms or used an explicit straight line constant velocity background [25]. The tools that we develop in this chapter will be later generalized to analyze multigraviton scattering.

2.1 Matrix Model low energy effective action for $SU(2)$

The matrix model Lagrangian is obtained from the dimensional reduction of $\mathcal{N} = 1$ supersymmetric Yang-Mills theory in $D = 9 + 1$ down to $D = 0 + 1$ dimensions [10]. For our purposes it will be useful to initially keep the action in its ten dimensional form expressed as

$$S = \int d^{10}x \left(-\frac{1}{4g} F_{\mu\nu}^a F^{\mu\nu a} + \frac{i}{2} \bar{\Psi}^a \Gamma^\mu D_\mu \Psi^a \right) \quad (2.1)$$

where the field strength is given by

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{abc} A_\mu^b A_\nu^c, \quad (2.2)$$

and the 32×32 dimensional Dirac matrices Γ satisfy the usual algebra $\{\Gamma_\mu, \Gamma_\nu\} = 2g_{\mu\nu}$ with metric $g_{\mu\nu} = \text{diag}(+1, -1, \dots, -1)$. The 32 component Majorana-Weyl adjoint spinor Ψ^a has only 16 real physical components off mass shell. We should mention that the center of mass motion of the $D0$ particles has been removed and we are considering the $SU(2)$ theory.

To calculate the one loop contributions to the effective action, we will use the background field method [26] and break the gauge field up into a classical background field and a fluctuating quantum field,

$$A_\mu^a \rightarrow X_\mu^a + A_\mu^{'a}, \quad (2.3)$$

and choose our gauge fixing condition, $D^\mu A_\mu^{'a} = 0$, to be covariant with respect to the background field, $D_\mu = \partial_\mu - it^a X_\mu^a$. Using the generators for Lorentz transformations on 4-vectors, $(\mathcal{J}^{\rho\sigma})_{\alpha\beta} = i(\delta_\alpha^\rho \delta_\beta^\sigma - \delta_\alpha^\sigma \delta_\beta^\rho)$, and spinors, $S^{\mu\nu} = \frac{i}{4}[\Gamma^\mu, \Gamma^\nu]$, and only keeping terms quadratic in the quantum fields, one obtains the gauge-fixed Lagrangian in the Feynman-

't Hooft gauge:

$$\mathcal{L} = \mathcal{L}_B + \mathcal{L}_{A'} + \mathcal{L}_\psi + \mathcal{L}_c. \quad (2.4)$$

The first piece of the Lagrangian just contains the background gauge field,

$$\mathcal{L}_B = -\frac{1}{4g} F_{\mu\nu}^a F^{\mu\nu a} \quad (2.5)$$

whereas the other pieces are quadratic in their respective quantum fields and contain the background gauge field in the background covariant derivative squared, D^2 , as well as in the background field strength $F_{\rho\sigma}^b$:

$$\mathcal{L}_{A'} = -\frac{1}{2g} \left\{ A_{\mu}^{'a} \left[-\left(D^2\right)^{ac} g^{\mu\nu} + \left(F_{\rho\sigma}^b \mathcal{J}^{\rho\sigma}\right)^{\mu\nu} \left(t^b\right)^{ac} \right] A_{\nu}^{'c} \right\} \quad (2.6)$$

$$\mathcal{L}_\psi = \frac{1}{2} \bar{\Psi}^a \left[\sqrt{-\left(D^2\right)^{ac} + \left(F_{\rho\sigma}^b S^{\rho\sigma}\right) \left(t^b\right)^{ac}} \right] \Psi^c \quad (2.7)$$

$$\mathcal{L}_c = \bar{c}^a \left[-\left(D^2\right)^{ab} \right] c^b. \quad (2.8)$$

In the fermion term, the square root arises from squaring $i\Gamma^\mu D_\mu$ and then taking the square root with the virtue of putting each of the quadratic parts of the Lagrangian in the same form. This will allow the supersymmetric cancellations of diagrams to become explicit without having to evaluate individual Feynman diagrams.

The one loop effective action is obtained by evaluating the functional integral for the quantum fields,

$$e^{i\Gamma[X]} = \int \mathcal{D}A' \mathcal{D}\Psi \mathcal{D}c \exp[i \int d^{10}x (\mathcal{L}_B + \mathcal{L}_{A'} + \mathcal{L}_\psi + \mathcal{L}_c)], \quad (2.9)$$

giving

$$\begin{aligned} \Gamma[X] = \int d^{10}x & \left(-\frac{1}{4g} F_{\mu\nu}^a F^{\mu\nu a} \right) + \frac{i}{2} \ln \text{Det} \left[-\left(D^2\right) g^{\mu\nu} + \left(F_{\rho\sigma}^b \mathcal{J}^{\rho\sigma}\right)^{\mu\nu} t^b \right] \\ & - \frac{i}{8} \ln \text{Det} \left[-\left(D^2\right) + \left(F_{\rho\sigma}^b S^{\rho\sigma}\right) t^b \right] - i \ln \text{Det} \left[-\left(D^2\right) \right]. \end{aligned} \quad (2.10)$$

For the fermion functional integration the extra factor $\frac{1}{4}$ arises from the fermion field having 16 real components instead of 32 complex ones.

To compute the determinants for the different fields, it is useful to expand D^2 ,

$$-D^2 = -\partial^2 + \Delta_1 + \Delta_2 \quad (2.11)$$

where

$$\Delta_1 = it^a (\partial_\mu X^{\mu a} + X_\mu^a \partial^\mu) \quad (2.12)$$

$$\Delta_2 = X_\mu^a t^a X^{\mu b} t^b. \quad (2.13)$$

At this point it is convenient to dimensionally reduce to 1-D while choosing $X_0^a = 0$, then $\Delta_1 = 0$. Choosing the $D0$ -branes to have a separation of r in the ninth transverse spatial direction, $X_\mu^a \rightarrow r\delta_3^a \delta_\mu^9 + X_\mu^a$, we can break $SU(2) \rightarrow U(1)$ giving

$$\Delta_2 = -r^2 t^3 t^3 - 2r X_9^a t^a t^3 - X_i^a t^a X^{ib} t^b \quad (2.14)$$

with the Latin index going 1–9 and fields X_i^a depending only on time. The magnetic moment interaction for the bosons

$$\Delta_J^B = \left(F_{\rho\sigma}^b \mathcal{J}^{\rho\sigma} \right)^{\mu\nu} t^b \quad (2.15)$$

dimensionally reduced becomes

$$\Delta_J^B = 2 \left(\partial_0 X_i^b \mathcal{J}^{0i} \right)^{\mu\nu} t^b \quad (2.16)$$

since we will be working in a flat direction. Similarly for the fermions one has

$$\Delta_J^\psi = 2 \left(\partial_0 X_i^b S^{0i} \right) t^b, \quad (2.17)$$

where we recognize $\partial_0 X_i = F_{0i}$ as the velocity. The general form of a determinant in (2.10) can be written

$$Tr \ln (-\partial_0^2 + \Delta_2 + \Delta_J). \quad (2.18)$$

Because we will only be considering the case of massless background fields we combine ∂^2 with r^2 and define

$$\Delta_F = \frac{1}{-\partial_0^2 - r^2} \quad (2.19)$$

in addition to

$$\Delta_b = -2r X_9^a t^a t^3 \quad (2.20)$$

$$\Delta'_2 = -X_i^a t^a X^{ib} t^b \quad (2.21)$$

then the trace becomes

$$Tr \ln(-\partial_0^2 - r^2) + Tr \ln[1 + \Delta_F(\Delta'_2 + \Delta_b + \Delta_J)]. \quad (2.22)$$

The first piece involving $-\partial_0^2 - r^2$ is a constant and the second contains the one loop quantum correction to the effective action which we will evaluate below by expanding the logarithm for various numbers of external background fields.

2.1.1 Non-Renormalization of F_{0i}^2

The various contributions with two external fields are,

$$Tr [\Delta_F \Delta'_2], \quad (2.23)$$

$$-\frac{1}{2} Tr [\Delta_F \Delta_b \Delta_F \Delta_b] \quad (2.24)$$

and

$$-\frac{1}{2} Tr [\Delta_F \Delta_J \Delta_F \Delta_J]. \quad (2.25)$$

The first and second terms occur in the determinant for the gauge, fermion and ghost fields whereas the last term only occurs for gauge and fermion fields. Writing out (2.23) in

frequency space gives

$$Tr[\Delta_F \Delta'_2] = -Tr[t^3 t^3] \int \frac{dw_1}{2\pi} X_i^3(w_1) X^{i3}(-w_1) \int \frac{dw}{2\pi} \frac{1}{(w^2 - r^2)} \quad (2.26)$$

which involves $Tr[t^3 t^3] = 2 d(j)$, where $d(j)$ is the number of components for the various fields

$$d(j)^\psi = 32 \quad d(j)^{A'} = 10 \quad d(j)^c = 1, \quad (2.27)$$

and the 2 arises from the trace of an $SU(2)$ generator squared in the adjoint representation. Now it becomes clear that identical terms which arise in each of the three determinants appearing in (2.10) will cancel. To be explicit one gets

$$[\frac{i}{2}(10) - \frac{i}{8}(32) - i(1)][Term] = 0 \quad (2.28)$$

where $[Term]$ is the frequency integral of (2.23) or (2.24). The supersymmetric cancellation of (2.25) between bosons and fermions is slightly more subtle requiring the determination of $Tr[S^{0i} S^{0j}] = 8g^{ij}$ for fermions and $Tr[\mathcal{J}^{oi} \mathcal{J}^{oj}] = 2g^{ij}$ for bosons. Putting the terms into (2.10) gives

$$[\frac{i}{2}(2) - \frac{i}{8}(8) - i(0)] \int \frac{dw_1}{2\pi} w_1^2 X_i^3(w_1) X^{i3}(-w_1) \int \frac{dw}{2\pi} \frac{1}{(w^2 - r^2)} \frac{1}{[(w - w_1)^2 - r^2]} = 0 \quad (2.29)$$

which shows that the 2-point contribution to the effective action at one-loop is zero. This result is consistent with the fact that $\mathcal{N} = 4$ super Yang-Mills in four dimensions receives no renormalizations of the kinetic terms.

2.1.2 Coefficient of v^4/r^7

With four external fields, the only term which doesn't cancel by the arguments given above is

$$-\frac{1}{4} Tr[\Delta_F \Delta_J \Delta_F \Delta_J \Delta_F \Delta_J \Delta_F \Delta_J] \quad (2.30)$$

or in frequency space

$$\int \frac{dw_2 dw_3 dw_4 dw - (w_2 + w_3 + w_4) X_i^3 [-(w_2 + w_3 + w_4)] w_2 X_j^3(w_2) w_3 X_k^3(w_3) w_4 X_l^3(w_4)}{(2\pi)^4 [(w + w_2)^2 - r^2] [(w + w_2 + w_3)^2 - r^2] [(w + w_2 + w_3 + w_4)^2 - r^2] [w^2 - r^2]} \quad (2.31)$$

with the prefactor

$$-\frac{1}{4} 2^4 \text{Tr}[(t^3)^4] \text{Tr}[(\mathcal{J}^{0i})^4] \quad (2.32)$$

for the gauge boson case. An identical result holds for fermions if one replaces the Lorentz generator trace with

$$\text{Tr}[S^{0i} S^{0j} S^{0k} S^{0l}] = 2(g^{ij} g^{kl} - g^{ik} g^{jl} + g^{il} g^{jk}) \quad (2.33)$$

whereas for the gauge bosons one finds

$$\text{Tr}[\mathcal{J}^{0i} \mathcal{J}^{0j} \mathcal{J}^{0k} \mathcal{J}^{0l}] = (g^{ij} g^{kl} + g^{il} g^{jk}). \quad (2.34)$$

Now using (2.10) and the low energy approximation $w_1, w_2, w_3, w_4 \rightarrow 0$, we get

$$-6i[(F_{0i}^3)^2]^2 \int \frac{dw}{2\pi} \frac{1}{(w^2 - r^2)^4}. \quad (2.35)$$

The integral can be performed in the complex plane using the usual $+i\epsilon$ prescription for handling the poles. One is left with the well known result [27, 25, 24]

$$\mathcal{L}_{eff}^{(1)} = \frac{15}{16} \frac{v_{12}^4}{r^7} \quad (2.36)$$

suppressing factors of N , R_{11} , M_{pl} and using $v_{12}^2 = (F_{oi}^3)^2 = (\vec{v}_1 - \vec{v}_2)^2$. Although many authors have obtained this same result, only the calculation discussed in [24] was sensitive to acceleration terms. Since the calculation in [24] lacked details, we wanted to see explicitly that the matrix model did not have acceleration terms in the one-loop effective action with four external fields as we have just demonstrated. We will see below that the leading term

in a $1/r$ expansion of the supergravity effective action for graviton-graviton scattering has precisely this structure and numeric coefficient.

2.2 Supergravity Effective Action: Two Graviton Interaction

Now that we have the leading term in the matrix model low energy effective action we are in a position to test the DLCQ matrix model-supergravity conjecture. In this section we will calculate the leading term in the effective potential between two gravitons and compare it with the matrix model calculation (2.36).

Since there are no couplings of two gravitons with a gravitino in the supergravity action, a tree-level calculation of graviton scattering only involves gravitons. It is therefore sufficient to proceed with the Feynman rules derived from the Einstein-Hilbert action in eleven dimensions,

$$S = \frac{1}{2\kappa^2} \int d^{11}x \sqrt{-g} R. \quad (2.37)$$

In (2.37), $\kappa^2 = 16\pi^5$ is the gravitational coupling constant using units with $M_{pl} = 1$, $g = \det(g_{\mu\nu})$ and R is the Ricci scalar curvature. To derive the Feynman rules for the graviton propagator and vertices one expands (2.37) in powers of the coupling by writing the metric as

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \quad (2.38)$$

where $\eta_{\mu\nu}$ is the flat space-time metric and the metric perturbation $h_{\mu\nu}$ is identified with the gravitational field. Such a procedure was first carried out by DeWitt [28]. The Feynman rules for the three and four-graviton vertex functions as well as the propagator in D space-time dimensions are nicely summarized in [29].

Graviton-graviton scattering at long distance or small momentum transfer, q , requires the evaluation of the Feynman diagram shown in figure 2.1. The incoming graviton

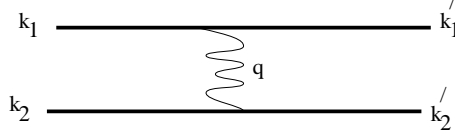


Figure 2.1: Graviton-graviton scattering at small momentum transfer. Solid lines are the scattering gravitons. The wiggly line represents a virtual graviton with zero longitudinal momentum.

momentum are k_1 and k_2 with outgoing momentum given by k'_1 and k'_2 . Each external on-shell graviton has a symmetric, traceless polarization tensor of the form, $\epsilon_{\mu\nu}$, which is transverse to the graviton propagation, $k^\mu \epsilon_{\mu\nu} = 0$. In evaluating this diagram, a number of simplifications occur by considering the $q \rightarrow 0$ limit and realizing that the leading term in the matrix model calculation (2.36) preserves the helicity of the scattering gravitons. In particular, $k_i \approx -k'_i$ (the Feynman rules in [29] define all momentum flowing into the vertex) or $k_i \cdot k'_i = 0$ ($i = 1$ or 2) and we are only interested in terms with the polarization tensors dotting into themselves, $(\epsilon_1 \cdot \epsilon'_1)(\epsilon_2 \cdot \epsilon'_2)$. Given these simplifications along with the fact that the external gravitons are traceless and transverse, we only need to consider two out of the eleven terms in the 3-vertex function¹

$$2\kappa \text{sym}[P_3(k_{1\sigma}k'_{1\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + P_6(k_{1\sigma}k_{1\gamma}\eta_{\mu\nu}\eta_{\alpha\beta})], \quad (2.39)$$

where sym means that the result must be symmetric in the three graviton indices, $\mu\alpha, \nu\beta, \sigma\gamma$ and P with the subscript indicates the number of distinct permutations of the momentum-graviton index combinations. As an example, consider the first term in (2.39),

$$\text{sym}P_3(k_{1\sigma}k'_{1\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) = k_{1\sigma}k'_{1\gamma}\eta_{\mu(\nu}\eta_{\alpha)\beta} + k_{1\nu}q_\beta\eta_{\mu(\sigma}\eta_{\alpha)\gamma} + k'_{1\mu}q_\alpha\eta_{\sigma(\nu}\eta_{\gamma)\beta} \quad (2.40)$$

¹To be consistent with the authors [30, 31, 24] who use $\kappa^2 = 16\pi^5$, the 3-vertex in [29] needs to be multiplied by 2.

with

$$\eta_{\mu(\nu}\eta_{\alpha)\beta} = \frac{1}{2}(\eta_{\mu\nu}\eta_{\alpha\beta} + \eta_{\mu\alpha}\eta_{\nu\beta}). \quad (2.41)$$

Performing the various contractions for the vertex involving graviton one and keeping the relevant terms gives, $2\kappa k_{1\sigma}k_{1\gamma}(\epsilon_1 \cdot \epsilon'_1)$. A similar result holds for the vertex involving graviton two, $2\kappa k_{2\sigma}k_{2\gamma}(\epsilon_2 \cdot \epsilon'_2)$. Thus contracting the 3-vertices together with a graviton propagator gives the term of interest for graviton-graviton scattering,

$$\frac{4\kappa^2(k_1 \cdot k_2)^2(\epsilon_1 \cdot \epsilon'_1)(\epsilon_2 \cdot \epsilon'_2)}{q^2}. \quad (2.42)$$

To compare (2.42) with (2.36) we need to convert to light cone variables with non-relativistic normalization.

Light cone variables can be defined in various ways depending on where one chooses to place factors of two. We use $k^+ = k_o + k_{11}$ and $k^- = k_o - k_{11}$ where $k_{11} = N/R_{11}$ carries the discrete Kaluza-Klein momentum. Then defining the invariant scalar product of two space-time vectors to be

$$k_1 \cdot k_2 = \frac{1}{2}(k_1^+k_2^- + k_1^-k_2^+) - \vec{k}_1 \cdot \vec{k}_2, \quad (2.43)$$

one can show that

$$k_1 \cdot k_2 = \frac{k_1^+k_2^+}{8}(\vec{v}_1 - \vec{v}_2)^2 \quad (2.44)$$

where $k^+ = 2k_{11}$ and we have used $\vec{k} = k_{11}\vec{v}$ along with $k^- = \vec{k}^2/2k_{11}$. To obtain the non-relativistic amplitude we divide by $\sqrt{k^+} = \sqrt{2E}$ for each external graviton giving

$$\frac{\kappa^2 N_1 N_2 v_{12}^4}{4R_{11}^2 \vec{q}^2} \quad (2.45)$$

for the non-relativistic graviton-graviton amplitude expressed in light cone variables with zero longitudinal momentum transfer, $q^+ = 0$, and $\vec{v}_{12} = \vec{v}_1 - \vec{v}_2$. Now taking the Fourier

transform of the transverse momentum to obtain the effective potential between gravitons, one arrives at

$$V_{eff}(r) = \frac{1}{2\pi R_{11}} \int \frac{d^9 q}{(2\pi)^9} \frac{\kappa^2 N_1 N_2 v_{12}^4}{4 R_{11}^2 \vec{q}^2} = \frac{15}{16} \frac{N_1 N_2}{R_{11}^3} \frac{v_{12}^4}{r^7} \quad (2.46)$$

in perfect agreement with the matrix model result (2.36). Of course one might argue that we have made the supergravity result agree with the matrix model by our choice of κ . However, as discussed in [24], the value of κ is determined by comparing the membrane tension when M-theory is compactified on a circle with the type IIA string tension. In addition, as we will see in chapter 4, this same value of κ gives agreement between supergravity and the matrix model for multigraviton scattering.

2.3 Importance of Non-Renormalization Theorems

As argued in the original proposal for the Matrix description of M-Theory [10] the agreement between supergravity and the matrix model that we have displayed in the previous sections can only be understood if the one-loop matrix model result, v^4/r^7 , is exact. To understand this point more clearly it is useful to write down the loop expansion in powers of v and r for the bosonic terms in the $SU(2)$ matrix model effective action [31],

$$\begin{aligned} \mathcal{L}_0 &= c_{00} v^2 \\ \mathcal{L}_1 &= c_{11} \frac{v^4}{r^7} + c_{12} \frac{v^6}{r^{11}} + c_{13} \frac{v^8}{r^{15}} + \dots \\ \mathcal{L}_2 &= c_{21} \frac{v^4}{r^{10}} + c_{22} \frac{v^6}{r^{14}} + c_{23} \frac{v^8}{r^{18}} + \dots \\ \mathcal{L}_3 &= c_{31} \frac{v^4}{r^{13}} + c_{32} \frac{v^6}{r^{17}} + c_{33} \frac{v^8}{r^{21}} + \dots \end{aligned} \quad (2.47)$$

The authors in [25] gave direct evidence that the two-loop correction to v^4/r^7 was zero by showing the coefficient $c_{21} = 0$. In the next chapter we will show that potential infrared

corrections to the four derivative term, v^4 , at arbitrary loop order cancel. Historically, this cancellation of infrared corrections to the v^4 term was the first evidence for the non-renormalization of v^4 at all orders in the matrix model. The complete proof for the matrix model was given shortly after in [32]. Using constraints from supersymmetry they were able to show that indeed the v^4/r^7 term is exact and the higher order coefficients c_{i1} at loop i should be zero.

Chapter 3

Evidence of a Non-Renormalization Theorem for v^4 terms

In this chapter, we will give additional evidence supporting the non-renormalization theorem for v^4 in low dimensions. We will also see that it is not possible to make a definite statement about v^6 with our techniques. In doing so, the first question we have to ask is: “non-renormalization of what?” In four dimensions we are used to the idea that non-renormalization theorems are statements about a Wilsonian effective action. For example, the non-renormalization theorem discussed in [33] is derived by considering the $\mathcal{N} = 4$ theory on its Coulomb branch, and studying the effective action obtained by integrating out massive and high frequency modes. In $0 + 1$ and $1 + 1$ dimensions (or in finite volume), however, there is not a notion of a moduli space in the same sense. Instead, one must adopt a Born-Oppenheimer treatment of the problem, thinking of holding the slow modes fixed and solving for the dynamics of the fast modes.

The approach of most authors has been to compute the one particle irreducible

effective action, using conventional field theory rules. Consider the case of $SU(2)$. As shown at the end of the previous chapter, at a given order in v , the loop expansion is formally an expansion in powers of $1/r^3$ ($1/r^2$), in $0+1$ ($1+1$) dimensions [31]. Recall, r is the expectation value of the adjoint fields (transverse separation of the gravitons, in the matrix model interpretation). The spectrum includes states with mass (frequency) of order r and massless states. In the two-loop computation of [25], individual diagrams contributing to the effective action containing massless states are infrared divergent. The authors of this reference dealt with this by using dimensional regularization, defining

$$\int \frac{d^d p}{p^2} = 0. \quad (3.1)$$

With this regulator, these authors find that there is no renormalization of the v^4 term. The result involves not only fermi-bose cancellations, but also cancellations between diagrams containing only massive states and diagrams containing massless states.

This result is encouraging, but since infrared divergences usually signal real physics, one might worry about the regularization procedure. However, there are many infrared divergent diagrams, and, as we will see in section 3.1, in the case of v^4 , the infrared divergences cancel and there is no sensitivity to the regularization procedure. We will also see that this cancellation is quite special to v^4 , and there is no reason to expect it to occur for higher orders in velocity.

While it is true that we do not have a good definition of a Wilsonian effective action, for the success of the naive DLCQ, what really interests us is the scattering amplitude. For the success of the naive DLCQ, at $\mathcal{O}(v^4)$, we actually require that there should be no corrections to this amplitude. This is, as we will explain in the next section, equivalent to the requirement that there should be no corrections to the 1PI effective action.

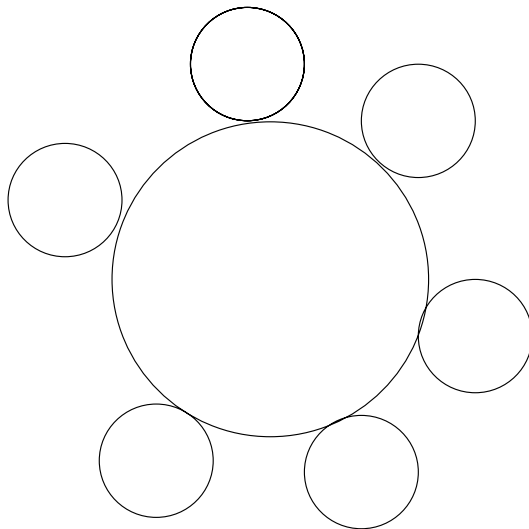


Figure 3.1: Infrared divergent contributions to the effective action.

The origin of the infrared problem is easily understood. In ℓ loop order, $\ell \geq 2$, consider the diagram shown in fig. 3.1. Here the central loop contains a massive field, and the $\ell - 1$ smaller loops contain massless fields. In momentum space, this graph is proportional to

$$\frac{1}{r^{6+2(\ell)-d}} \left(\int \frac{d^d p}{p^2} \right)^{\ell-1}. \quad (3.2)$$

Alternatively, if the amplitudes are written in coordinate space, the propagator is ambiguous; individual diagrams are proportional to this ambiguity. In the infrared limit, one can think of the integral over the massive states as generating a local operator, and the massless integrals as giving the “vacuum matrix element” of this operator. This same type of analysis can be performed for all of the infrared divergent graphs. For the v^4 terms, we will see in the next section that this matrix element vanishes. However, this cancellation depends crucially on the fact that $1/r^7$ is the Green’s function for the nine-dimensional Laplace operator, and might not hold for higher powers of v . v^6 turns out to be special

as well, because the one loop contribution vanishes [31]¹, and we cannot make a definite statement.

In the case of $SU(3)$ (and higher rank groups), one can also exhibit the cancellation of certain finite renormalizations. In this case there are two (or more) scales, R and r . As in [35], one can consider a hierarchy of scales (impact parameters, in the matrix model interpretation), $R \gg r$. Again, the diagrams contributing to the effective action contain infrared divergent terms. But there are also finite terms which behave as $(1/R^2 r)^{\ell-1}$. It is easy to isolate these terms. Diagrams such as those of fig. 3.1, where now the small loops contain fields of mass r and the big loop masses of order R , are of the form

$$\frac{v^4}{R^{6+2(\ell)-d}} \left(\int \frac{d^d p}{p^2 + r^2} \right)^{\ell-1} \sim \frac{v^4}{R^{6+2\ell-d} r^{\ell-d}} \quad (3.3)$$

(in $1+1$, the r dependence is logarithmic). In section 3.2, we will see that there is a cancellation of the most singular term at order v^4 for $\ell = 2$. Based on the results for $SU(2)$, it seems quite plausible that this cancellation persists to all orders. From the perspective of the matrix model, this is reassuring, since there would be no sensible spacetime interpretation for such terms. As for $SU(2)$, it is not easy to decide what happens at order v^6 .

However, to determine the full implications of these results requires settling some subtle issues. In particular, for these low dimension theories, the significance of the effective action is not completely clear, obscured, as we have noted, by infrared and (related²) operator ordering questions. We will offer some remarks on these issues, but will not completely resolve them.

¹This is only true in the straight line scattering approximation, for more general backgrounds v^6 is non-zero and is accompanied by acceleration terms [34].

²We thank Nathan Seiberg for stressing this connection to us.

3.1 Infrared Divergences in $SU(2)$

Consider, first, the matrix model with $N = 2$ in $d = 0 + 1$. We will write the bosonic part in terms of a set of “fields,” x^i , $i = 1, \dots, 9$, and a “gauge boson,” A . All of these fields are $SU(2)$ matrices. There are flat directions with \vec{x} a diagonal matrix,

$$\vec{x} = r\vec{\tau}_3/2. \quad (3.4)$$

Correspondingly, there are a set of massive modes (i.e. modes with frequencies proportional to r) and massless modes. At one loop, integrating out the massive modes in this model is well-known to generate an effective action, whose leading bosonic term was calculated in the previous chapter to be

$$\mathcal{L}_{eff}^{(1)} = \frac{15}{16} \frac{v^4}{r^7}. \quad (3.5)$$

When considering the scattering amplitude, in a path integral approach, one is interested in

$$\langle \vec{x}_f(t_f) | \vec{x}_i(t_i) \rangle \quad (3.6)$$

where \vec{x}_f and \vec{x}_i are the eigenvalues of \vec{x}_3 , the diagonal component of the matrix. Expanding \vec{x} about the classical solution

$$\vec{x}(t) = \vec{x}_{cl} + \delta\vec{x} \quad (3.7)$$

$$\begin{aligned} \vec{x}_i + \frac{\vec{x}_f - \vec{x}_i}{t_f - t_i}(t - t_i) + \delta\vec{x}(t) \\ = \vec{b} + \vec{v}t + \delta\vec{x} \end{aligned}$$

one studies the region of large $|\vec{b}|$, small $|\vec{v}|$. In this regime, the amplitude can be expanded in powers of \vec{v} [27, 20]. At higher orders, as we have noted, there is a serious potential

for infrared divergences. In $1 + 1$ dimension, the problem is familiar from string theory.

Written in a fourier decomposition, the two dimensional massless propagator is:

$$\langle x(\sigma)x(\sigma') \rangle = \int d^2k \frac{e^{ik \cdot (\sigma - \sigma')}}{k^2}, \quad (3.8)$$

which is ill defined. Correspondingly, the coordinate space expression is

$$\langle x(\sigma)x(\sigma') \rangle = \ln(\sigma - \sigma')^2 + \text{constant}. \quad (3.9)$$

In string theory, one only considers Green's functions of translationally invariant combinations of operators, and these are infrared finite; equivalently, they are independent of the arbitrary constant.

In $0 + 1$ dimensions, the divergences are even more severe. If we try to write a momentum (frequency) space propagator we have

$$\langle \delta x_i(t) \delta x_j(t') \rangle = \delta_{ij} \int d\omega \frac{e^{-i\omega(t-t')}}{\omega^2} \quad (3.10)$$

which is linearly divergent. Correspondingly, the coordinate space Green's function is ambiguous (dropping the vector symbol):

$$\langle \delta x(t) \delta x(0) \rangle = at\theta(t) - bt\theta(-t) + ct + d, \quad (3.11)$$

with $a + b = 1$. The authors in [25] assumed that the coefficients b , c and d are zero, however, recent work by Rong Li suggests that the equal time propagator should vanish in order to reproduce the Born series in potential scattering [36].

When we say below that infrared divergences do (or do not) cancel in $1 + 1$ or $0 + 1$ dimension, we will mean that they cancel at the level of momentum space expansions, or alternatively that the quantities in question are not sensitive to the ambiguities in the propagators.

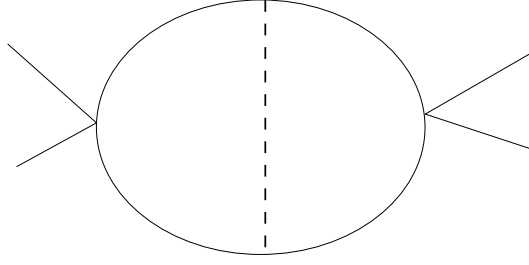


Figure 3.2: Some two loop corrections to the effective action.

Now consider two loop corrections to the effective action. Some sample diagrams are shown in fig. 3.2. Consider, in particular, diagrams with one massive state and one massless state running in the loop. Individual diagrams with massless states in the loop are infrared divergent, behaving as

$$\int \frac{d\omega}{\omega^2} \quad (3.12)$$

for small frequencies. Note that the external x 's must always attach to massive lines. Because of this fact, and because the leading infrared divergence always comes from such a small frequency region of integration, the leading divergent piece of each diagram always factorizes into a product of two one loop terms. One is a massive loop, with four external “scalars” (x 's), on which the time derivatives act, and two more without derivatives. The two without derivatives are then contracted with each other, forming the massless loop. In other words, the infrared divergent terms can all be organized in terms of operators generated at one loop of the form

$$\mathcal{O} = v^4 \delta x^2 / r^9. \quad (3.13)$$

The infrared divergence then arises from simply contracting the two factors of x in this expression, i.e. taking the “vacuum matrix element.”

However, we do not need to compute all of the diagrams to determine the coefficient of this term in the effective action! In eqn. 5, we can interpret r^2 as $(\vec{x}_{cl} + \delta\vec{x})^2$, and expand in powers of $\delta\vec{x}$. This gives

$$\mathcal{O}_1 = \frac{v^4}{x_{cl}^7} \left(1 - \frac{7}{2x_{cl}^2} (2\vec{x}_{cl} \cdot \vec{\delta x} + \vec{\delta x}^2) + \frac{7 \times 9}{8x_{cl}^4} (2\vec{x}_{cl} \cdot \vec{\delta x} + \vec{\delta x}^2)^2 \right). \quad (3.14)$$

Taking the expectation value, the last two (infrared divergent) terms in this expression cancel because there are nine x 's. A similar cancellation occurs in $1 + 1$ dimension.

It should be noted that there are no potential infrared divergences from other diagrams. Diagrams involving gauge fields (which exist in gauges other than $A^0 = 0$) are not divergent. The one loop effective action must be gauge invariant, and this means that it must be independent of the gauge field in $0 + 1$ dimensions, and involve at least two time derivatives in $1 + 1$ dimension. Diagrams involving fermions are not as divergent due to the structure of the fermion propagator and have the wrong scaling with r .

It is easy to extend this argument for the cancellation of the most infrared singular terms to higher orders. At each order, the most singular contribution comes from diagrams where several massless scalars attach to a single loop of massive fields. These diagrams correspond to expanding the $1/r^7$ term to higher orders in x , and contracting the x 's. But $1/r^7$ is special, as it is the Green's function for the nine-dimensional laplacian. This means that, for $r \neq 0$, $r \gg \delta x$,

$$\nabla^2 \frac{1}{|\vec{r} + \delta\vec{x}|^7} = 0 \quad (3.15)$$

where the derivatives act with respect to \vec{r} . Expanding in powers of $\delta\vec{x}$, this must be true for every term in the sum. It must also, then, be true when we average over $\delta\vec{x}$. But averaged over $\delta\vec{x}$, each term is proportional to $\frac{1}{r^n}$ (times an infrared divergent integral). So, except

for the leading term, the coefficient of every other term in the expansion must vanish, upon averaging. The skeptical reader is invited to check the next order explicitly.

Note that in the path integral framework, the non-renormalization of the v^4 terms (and the cancellation of ir divergences) in the effective action immediately implies the same for the scattering amplitude. It is important to note that the terms in the supersymmetric completion of the v^4 term can each be written in the $1/r^7$ form [37] (see e.g. chapter five equation 5.2) and the argument given above applies to them as well.

Now consider higher orders in velocity. At one loop, there is no v^6 term in the effective action. There is a v^8 term,

$$\frac{v^8}{r^{15}}. \quad (3.16)$$

Expanding the denominator as before, one now finds that there is an apparent infrared divergence at two loops. However, we need to be careful of addition tensor structure which would also contribute to such a divergence. For example, a term of the form

$$\frac{v^6(\vec{v} \cdot \vec{r})^2}{r^{17}} \quad (3.17)$$

would contribute. Since the full tensor structure has not been calculated for v^8 , it is impossible to conclude if the v^8 term receives an infrared renormalization. The lesson to be learned is that any terms which scale like $\frac{1}{r^n}$ with $n \neq 7$ will have the potential to receive infrared renormalizations. Such a term will arise in chapter 4 when we consider multigraviton scattering.

Returning to the v^6 terms, as noted above, a v^6 term is not generated at one loop. Such a term is generated at two loops [31]. But we cannot simply apply our reasoning to the two loop case. The calculation of [31] includes graphs with both massive and massless

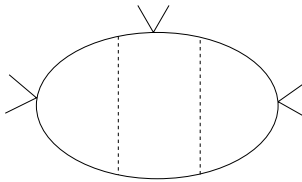


Figure 3.3: Three loop correction to the effective action.

states. At three loops, there are diagrams with zero, one or two massless particles in the loop. Expanding the two loop action in powers of $\delta\vec{x}$, and contracting $\langle \delta\vec{x}\delta\vec{x} \rangle$ correctly reproduces the infrared parts of diagrams with one massless field, but double counts the diagrams with two (see fig. 3.3). So we cannot establish by this means whether there is an infrared divergence (and a breakdown of the non-renormalization theorem) for $SU(2)$ at v^6 . This is just as well. The fact that the calculation of [31] successfully reproduces the naive DLCQ strongly suggests that there is a non-renormalization theorem for this case.

Finally, we should note that the authors of [25] have computed, using their regulator, the coefficient of the v^8 term at two loops [38]. However, they are not able to perform a direct comparison with supergravity.

3.2 Finite Renormalizations in $SU(3)$

Consider, now, an $SU(3)$ gauge group. In this case, taking x to be a $U(3)$ field, we will consider “expectation values” of the fields x of the form:

$$x^9 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & R \end{pmatrix}. \quad (3.18)$$

This is not the most general expectation value, but it is sufficient for our purposes. In the language of M theory or $D0$ branes, this corresponds to three gravitons (branes) at

locations $0, r$ and R respectively.

Suppose that $r \ll R$. Then there is an approximate $SU(2)$ ($U(2)$) symmetry. We can then imagine first integrating out states with mass of order R , and then those with mass of order r , to obtain an effective action for the massless fields. At the first step, we expect to generate an operator of the form

$$\mathcal{O}_3 = \frac{\vec{v}_3^4 (a \vec{x}^a \cdot \vec{x}^a + b \hat{R} \cdot \vec{x}^a \hat{R} \cdot \vec{x}^a)}{R^9}, \quad (3.19)$$

where \vec{x}_a are the $SU(2)$ triplet fields. Then, replacing $x^a x^a$ by

$$\langle x_i^a x_j^a \rangle = \delta_{ij} \left(\frac{1}{r} + \int \frac{d\omega}{(2\pi)\omega^2} \right) \quad (3.20)$$

in this expression, we obtain a result proportional to

$$\mathcal{O}_4 = \frac{v_3^4}{R^9 r}, \quad (3.21)$$

as well as a potentially infrared divergent term.

As in the case of $SU(2)$, it is not difficult to verify that the coefficient of \mathcal{O}_4 , as well as the infrared divergence, vanishes to two loops. In the $SU(3)$ case, the one-loop result is:

$$\mathcal{L}_{v^4} \propto \left(\frac{v_{12}^4}{x_{12}^7} + \frac{v_{13}^4}{x_{13}^7} + \frac{v_{23}^4}{x_{23}^7} \right). \quad (3.22)$$

Write

$$\vec{x}_1 = \vec{x}_1 \quad \vec{x}_2 = \vec{r} + \vec{x}_2 \quad \vec{x}_3 = \vec{R} + \vec{x}_3, \quad (3.23)$$

and expand the last two terms in powers of the fluctuations, \vec{x} , keeping only the part proportional to v_3^4 . The first order terms are $SU(2)$ singlets (they are proportional to

$\vec{x}_1 + \vec{x}_2$). The quadratic terms contain the $SU(2)$ non-singlet fields, $\vec{x}_1 - \vec{x}_2$. These couplings can be generalized to the $SU(2)$ invariant coupling,

$$\delta\mathcal{L} \propto \left(\frac{v_3^4 x^a x^a}{R^9} - 9 \frac{v_3^4 (x^a \cdot R)^2}{R^{11}} \right). \quad (3.24)$$

Taking the expectation value, we see that as in the case of the $SU(2)$ infrared divergences, the leading $1/r$ and infrared divergent pieces cancel. It is not so easy to check higher orders, in this case, since one can't generalize, e.g., the \vec{x}^n terms unambiguously to $SU(2)$ -invariant expressions. However, we have checked explicitly the cancellation to next order, and expect the same will occur for higher orders.

Again, because of the vanishing of the v^6 term at one loop, we cannot establish by this sort of reasoning whether or not there are corrections to the various v^6 operators at three loops.

This argument can be extended to $1 + 1$ dimensions. There is again no infrared problem at $\mathcal{O}(v^4)$, and no terms which depend on $\ln(r)$ (the analog of the $1/r$ terms in the $0 + 1$ dimensional case).

Chapter 4

Multigraviton Scattering

The first test to see if the matrix model could reproduce multigraviton scattering in supergravity was performed by [35]. In [35] it was argued that there was a discrepancy between the computation of three graviton scattering in the matrix model and in tree level supergravity. Calling the large distance R and the smaller distance r , and denoting the velocity of the far-away graviton by v_3 , the supergravity S -matrix contains a term (after Fourier transform):

$$\frac{v_3^4 v_{12}^2}{r^7 R^7}. \tag{4.1}$$

However, we will show with a detailed calculation in section 4.2 that no such term can be generated in the matrix model *effective action*. The authors of [35] then went on to argue that this term could not appear in the Matrix model *S-matrix*.

Subsequently, however, Taylor and Van Raamsdonk [39] pointed out, using simple symmetry considerations, that if one writes an effective action for gravitons in supergravity, this action *cannot* contain such terms. Shortly afterwards, Okawa and Yoneya [30] computed the effective action on both the matrix model and supergravity sides, and showed that there

is complete agreement. A related computation appeared in [40]. Other calculations have also been reported recently showing impressive agreement between the matrix model and supergravity [41].

It is clear from these remarks that the difficulty in [35] lies in extracting the Matrix model S -matrix from the effective action. In section 4.4 we show how the “missing term” is generated in the S -matrix of the matrix model. In order to do this using the effective action approach, it is necessary to resolve certain operator-ordering questions¹. To deal with these issues the most efficient approach is the path integral. In section 4.4.1, we review first the problem of computing the S -matrix from the path integral by studying small fluctuations about classical trajectories. Once this is done, the isolation of the “missing term” is not difficult.

Despite the error in the analysis of [35], the method proposed there yields a considerable simplification in the calculation of the effective action. Indeed, it is possible to calculate certain terms in just a few lines. On the supergravity side, there are also significant simplifications which occur in this limit. One might hope, then, to extract general lessons from this approach. For example, one can compare certain tensor structures in n -graviton scattering, and perhaps try to understand whether (and why) there is agreement. One can also try to examine, as in [42] the role of non-renormalization theorems.

In section 4.5, then, we go on to compare certain other terms in three graviton scattering, some of which were not explicitly studied in [30]. These calculations can be performed using the methods proposed in [35], on both the matrix model and supergravity sides, and are shown to agree.

¹The authors of [35] had convinced themselves that there was no choice of operator ordering which generated the missing term. This was their basic error.

Armed with this success, we consider in section 4.6 scattering of more than three gravitons, and scattering when more dimensions are compactified. Some of the terms in the four graviton scattering amplitude can readily be computed, and compared on both sides. We find agreement of certain terms involving eight powers of velocity. We also find certain terms of order v^{2n} in n -graviton scattering, for arbitrary n , agree. On the other hand, the matrix model at three loops generates terms of order v^6 in four graviton scattering. These do not have the correct scaling with N to generate a Lorentz invariant expression, and it is difficult to see how they can be cancelled by other matrix model contributions to the S -matrix. These terms also indicate that there are terms at order v^6 which are renormalized.

These observations raise a number of questions. In particular, it is not completely clear why the arguments of [18] and [19] imply that the *classical* supergravity amplitudes should agree with the matrix model result. One might have thought that this should only hold in cases in which there are non-renormalization theorems [12]. Our results indicate that already at the level of the four graviton amplitude, there are not non-renormalization theorems, at least in the most naive sense. They also suggest that at order v^{2n} , the $n - 1$ loop matrix model diagram reproduces the supergravity amplitude, but that there are discrepancies at three loops and beyond in terms with fewer powers of velocity. We will make some remarks on these issues in the section 4.7, but will not provide a definite resolution.

4.1 Background

It is worthwhile to review the problems which arise when one tries to compare three graviton scattering in the matrix model picture with supergravity, setting the stage

for our notation which will be used below. Briefly, the authors of [35] considered the case of three gravitons; two separated a distance r from each other and another a distance R from the other two in the limit $R \gg r$. A term in the supergravity S -matrix for three graviton scattering in the small momentum transfer limit was shown to be

$$\frac{(k_1 \cdot k_2)(k_1 \cdot k_3)(k_2 \cdot k_3)}{q_1^2 q_2^2} \quad (4.2)$$

where k_i are the i th graviton momenta and $q_{1,2}$ are the two relevant momenta transfer. In the language of matrix theory, this corresponds to taking the Fourier transform of the two-loop effective potential

$$\frac{v_{12}^2 v_{13}^2 v_{23}^2}{R^7 r^7} \quad (4.3)$$

where $v_{12} = (v_1 - v_2)$, etc. refer to the relative velocities of the $D0$ -branes. The two scales R and r arise from integrating out the massive degrees of freedom introduced by giving the diagonal generators of $SU(3)$ vacuum expectation values:

$$\langle X_i^a \rangle = r \delta^{a3} \delta_{i1} + R \delta^{a8} \delta_{i2} \quad (4.4)$$

where X_i^a are the 9 $SU(3)$ -valued fields describing the bosonic coordinates. Since $\dot{X}_i = \dot{X}_i^8 T^8 + \dot{X}_i^3 T^3$, one can work out v_{12}^2 , etc. in terms of \dot{X}_i^3 and \dot{X}_i^8

$$v_{23}^2 \sim (\dot{X}_i^3)^2 \quad (4.5)$$

$$v_{13}^2 \sim (\dot{X}_i^3)^2 + (3\dot{X}_i^8)^2 - 6\dot{X}_i^8 \dot{X}_i^3 \quad (4.6)$$

$$v_{12}^2 \sim (\dot{X}_i^3)^2 + (3\dot{X}_i^8)^2 + 6\dot{X}_i^8 \dot{X}_i^3 \quad (4.7)$$

Multiplying these three together yields the expected result for matrix theory

$$v_{12}^2 v_{13}^2 v_{23}^2 \sim (\dot{X}_i^3)^2 (\dot{X}_i^8)^4 + (\dot{X}_i^3)^6 + (\dot{X}_i^3)^4 (\dot{X}_i^8)^2 - (\dot{X}_i^3 \dot{X}_i^8)^2 (\dot{X}_i^3)^2. \quad (4.8)$$

In [35] it was argued that matrix theory was incapable of reproducing the term,

$$\frac{(\dot{X}_i^8)^4(\dot{X}_i^3)^2}{R^7 r^7} \quad (4.9)$$

with the correct powers of R and r at two-loops. In [43], it was argued that this term can arise at two-loops from vertices with three massive bosons in the form of the setting-sun diagram, as well as from other two-loop interactions. After describing the background field method for $SU(3)$ below, we go on to show that the one-loop effective operator needed to arrive at the conclusion of [43] does indeed cancel among bosons and fermions. By exploiting the fact that \dot{X}_i^8 only couples to fields of scale R , we integrate out these most massive modes to find that the first term containing coupling between the heavy and light states without supersymmetric cancellations has the form $(\dot{X}_i^8)^4(X_i^a)^2/R^9$ as described in [35]. Then integrating over the light $SU(2)$ modes of scale r ($a=1,2$), we demonstrate that the term in the matrix model effective action with four powers of \dot{X}_i^8 and the least suppression in R is $(\dot{X}_i^8)^4(\dot{X}_i^3)^2/R^9 r^5$.

4.2 Matrix Model low energy effective action for $SU(3)$

The calculations performed in this section are a generalization of the calculations carried out in section 2.1 to the larger rank gauge group $SU(3)$. We will repeat the necessary background information to make this section self contained. For the reader who is familiar with material presented in section 2.1, the new content in this section begins just below equation (4.24).

The matrix model Lagrangian is obtained from the dimensional reduction of $\mathcal{N} = 1$ supersymmetric Yang-Mills theory in $D = 9 + 1$ down to $D = 0 + 1$ dimensions [10]. For our

purposes it will be useful to initially keep the action in its ten dimensional form expressed as

$$S = \int d^{10}x \left(-\frac{1}{4g} F_{\mu\nu}^a F^{\mu\nu a} + \frac{i}{2} \bar{\Psi}^a \Gamma^\mu D_\mu \Psi^a \right) \quad (4.10)$$

where the field strength is given by

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{abc} A_\mu^b A_\nu^c, \quad (4.11)$$

and the 32×32 dimensional Dirac matrices Γ satisfy the usual algebra $\{\Gamma_\mu, \Gamma_\nu\} = 2g_{\mu\nu}$ with metric $g_{\mu\nu} = \text{diag}(+1, -1, \dots, -1)$. The 32 component Majorana-Weyl adjoint spinor Ψ^a has only 16 real physical components off mass shell. We should mention that the center of mass motion of the $D0$ particles has been removed and we will be considering the $SU(3)$ theory with the gauge index $a=1-8$.

To calculate the one loop contributions to the effective action, we will use the background field method [26] and break the gauge field up into a classical background field and a fluctuating quantum field,

$$A_\mu^a \rightarrow X_\mu^a + A_\mu^{'a}, \quad (4.12)$$

and choose our gauge fixing condition, $D^\mu A_\mu^{'a} = 0$, to be covariant with respect to the background field, $D_\mu = \partial_\mu - it^a X_\mu^a$. By only keeping terms quadratic in the quantum fields, one obtains the gauge-fixed Lagrangian in the Feynman-'t Hooft gauge:

$$\mathcal{L} = \mathcal{L}_B + \mathcal{L}_{A'} + \mathcal{L}_\psi + \mathcal{L}_c. \quad (4.13)$$

The first piece of the Lagrangian just contains the background gauge field,

$$\mathcal{L}_B = -\frac{1}{4g} F_{\mu\nu}^a F^{\mu\nu a} \quad (4.14)$$

whereas the other pieces are quadratic in their respective quantum fields and contain the background gauge field in the background covariant derivative squared, D^2 , as well as in the background field strength $F_{\rho\sigma}^b$:

$$\mathcal{L}_{A'} = -\frac{1}{2g} \left\{ A_{\mu}^{'a} \left[-\left(D^2\right)^{ac} g^{\mu\nu} + \left(F_{\rho\sigma}^b \mathcal{J}^{\rho\sigma}\right)^{\mu\nu} \left(t^b\right)^{ac} \right] A_{\nu}^{'c} \right\} \quad (4.15)$$

$$\mathcal{L}_{\psi} = \frac{1}{2} \bar{\Psi}^a \left[\sqrt{-\left(D^2\right)^{ac} + \left(F_{\rho\sigma}^b S^{\rho\sigma}\right) \left(t^b\right)^{ac}} \right] \Psi^c \quad (4.16)$$

$$\mathcal{L}_c = \bar{c}^a \left[-\left(D^2\right)^{ab} \right] c^b \quad (4.17)$$

where

$$\left(\mathcal{J}^{\rho\sigma}\right)_{\alpha\beta} = i \left(\delta_{\alpha}^{\rho} \delta_{\beta}^{\sigma} - \delta_{\alpha}^{\sigma} \delta_{\beta}^{\rho} \right) \quad (4.18)$$

$$S^{\mu\nu} = \frac{i}{4} [\Gamma^{\mu}, \Gamma^{\nu}]. \quad (4.19)$$

The one loop effective action is obtained by evaluating the functional integral for the quantum fields,

$$e^{i\Gamma[X]} = \int \mathcal{D}A' \mathcal{D}\bar{\Psi} \mathcal{D}\Psi \mathcal{D}\bar{c} \mathcal{D}c \exp[i \int d^{10}x (\mathcal{L}_B + \mathcal{L}_{A'} + \mathcal{L}_{\psi} + \mathcal{L}_c)], \quad (4.20)$$

giving

$$\begin{aligned} \Gamma[X] = \int d^{10}x & \left(-\frac{1}{4g} F_{\mu\nu}^a F^{\mu\nu a} \right) + \frac{i}{2} \ln \text{Det} \left[-\left(D^2\right) g^{\mu\nu} + \left(F_{\rho\sigma}^b \mathcal{J}^{\rho\sigma}\right)^{\mu\nu} t^b \right] \\ & - \frac{i}{8} \ln \text{Det} \left[-\left(D^2\right) + \left(F_{\rho\sigma}^b S^{\rho\sigma}\right) t^b \right] - i \ln \text{Det} \left[-\left(D^2\right) \right]. \end{aligned} \quad (4.21)$$

For the fermion functional integration the extra factor $\frac{1}{4}$ arises from the fermion field having 16 real components instead of 32 complex ones.

To compute the determinants for the different fields, it is useful to expand D^2 ,

$$-D^2 = -\partial^2 + \triangle_1 + \triangle_2 \quad (4.22)$$

where

$$\Delta_1 = it^a (\partial_\mu X^{\mu a} + X_\mu^a \partial^\mu) \quad (4.23)$$

$$\Delta_2 = X_\mu^a t^a X^{\mu b} t^b. \quad (4.24)$$

At this point it is convenient to dimensionally reduce to 1-D while choosing $X_0^a = 0$, so $\Delta_1 = 0$. By letting $X_\mu^a \rightarrow r\delta_3^a\delta_\mu^1 + R\delta_8^a\delta_\mu^2 + X_\mu^a$ we can break $SU(3) \rightarrow U(1) \times U(1)$ giving

$$\Delta_2 = -r^2 t^3 t^3 - 2r X_1^a t^a t^3 - R^2 t^8 t^8 - 2R X_2^a t^a t^8 - X_i^a t^a X^{ib} t^b \quad (4.25)$$

with the Latin index going 1-9 and fields X_i^a depending only on time. It is important to note that in 1-D, r and R are dynamical variables and we are holding them fixed in the spirit of doing a Born-Oppenheimer approximation. The magnetic moment interaction for the bosons

$$\Delta_J^B = \left(F_{\rho\sigma}^b \mathcal{J}^{\rho\sigma} \right)^{\mu\nu} t^b \quad (4.26)$$

dimensionally reduced becomes

$$\Delta_J^B = 2 \left(\partial_0 X_i^b \mathcal{J}^{0i} \right)^{\mu\nu} t^b \quad (4.27)$$

since we will be working in a flat direction. Similarly for the fermions one has

$$\Delta_J^\psi = 2 \left(\partial_0 X_i^b S^{0i} \right) t^b. \quad (4.28)$$

The general form of a determinant in (4.21) can be written

$$Tr \ln(-\partial_0^2 + \Delta_2 + \Delta_J). \quad (4.29)$$

Because we are interested in the limit $R \gg r$ and will be letting only the most massive modes (scale R) run in the loop (gauge index a=4-7) then $t^3 t^3 r^2 = \frac{1}{4} r^2$ and $t^8 t^8 R^2 = \frac{3}{4} R^2$.

It is convenient to rescale, $r \rightarrow 2r$, $R \rightarrow \frac{2}{\sqrt{3}}R$ and define

$$\Delta_F = \frac{1}{-\partial_0^2 - R^2 - r^2} \quad (4.30)$$

in addition to

$$\Delta_r = -4r X_1^a t^a t^3 \quad (4.31)$$

$$\Delta_R = -\frac{4}{\sqrt{3}} R X_2^a t^a t^8 \quad (4.32)$$

$$\Delta'_2 = -X_i^a t^a X^{ib} t^b \quad (4.33)$$

then the trace becomes

$$Tr \ln(-\partial_0^2 - R^2 - r^2) + Tr \ln[1 + \Delta_F(\Delta'_2 + \Delta_r + \Delta_R + \Delta_J)]. \quad (4.34)$$

The first piece involving $-\partial_0^2 - R^2 - r^2$ is a constant and the second contains the one loop quantum corrections to the effective action which we will evaluate below by expanding the logarithm for various numbers of external background fields. We will find that the first non-zero terms contain four derivatives even if one just integrates over the most massive modes, R .

4.2.1 Terms with no derivatives

We will display in this section a supersymmetric cancellation between bosons and fermions for all operators which can be constructed from $-D^2$. Even before considering the expansion of $-D^2$ in (4.22), it is straightforward to see that all terms in the one loop effective action with no derivatives cancel. This is because the determinants of the bosons and fermions differ only by derivative terms, and there are an equal number of bosonic and fermionic factors in the determinant. Given that a non-derivative operator is particularly

important in the analysis of [43], we show explicitly in this section how non-derivative operators are cancelled.

The operator in question has the form

$$\delta\mathcal{L}_{eff}^{(1)} = \frac{r^2}{R^3} x_1^b x_1^b \quad (4.35)$$

where the gauge index, $b=1-2$, for the small mass $SU(2)$ subgroup (scale r). Such a term arises from expanding the logarithm in (4.34) and is given by

$$-\frac{1}{2}Tr [\Delta_F \Delta_r \Delta_F \Delta_r] \quad (4.36)$$

or in frequency space

$$-8Tr[t^3 t^a t^3 t^b] r^2 \int \frac{dw_1}{2\pi} x_1^a(w_1) x_1^b(-w_1) \int \frac{dw}{2\pi} \frac{1}{(w^2 - R^2)} \frac{1}{[(w - w_1)^2 - R^2]} \quad (4.37)$$

where we have dropped r^2 in Δ_F for the leading $1/R$ behavior. Integrating (4.37) in the limit $w_1 \rightarrow 0$ and then Fourier transforming gives (4.35). Now the important point to notice is that Δ_r arises from $-D^2$ which occurs in each determinant for the gauge, fermion and ghost fields (4.21). However, they each give a different contribution to $Tr[t^3 t^a t^3 t^b] \sim \delta^{ab} d(j)$, where $d(j)$ is the number of components for the various fields

$$d(j)^\psi = 32 \quad d(j)^{A'} = 10 \quad d(j)^c = 1. \quad (4.38)$$

Now it becomes clear that all terms coming from $-D^2$ in each of the three determinants appearing in (4.21) will cancel. To be explicit one gets

$$[\frac{i}{2}(10) - \frac{i}{8}(32) - i(1)][\frac{r^2}{R^3} x_1^b x_1^b] = 0. \quad (4.39)$$

A similar result holds for any number of external fields without derivatives involving Δ_r , Δ_R , and Δ'_2 .

4.2.2 Cancellation of $(F_{0i}^a)^2$ or $(\dot{X}_i^a)^2$

In this section, we show that terms with two derivatives cancel as well. This result is familiar in higher dimensions, where it is well known that the kinetic terms of the fields are not renormalized.

Based on the arguments given above the only possible non-vanishing term with two external fields contains two derivatives and is given by

$$-\frac{1}{2}Tr[\Delta_F\Delta_J\Delta_F\Delta_J]. \quad (4.40)$$

The supersymmetric cancellation of (4.40) between bosons and fermions requires the determination of $Tr[S^{0i}S^{0j}] = 8g^{ij}$ for fermions and $Tr[\mathcal{J}^{0i}\mathcal{J}^{0j}] = 2g^{ij}$ for bosons. Putting the term into (4.21) gives

$$\left[\frac{i}{2}(2) - \frac{i}{8}(8) - i(0)\right]Tr[t^a t^b] \int \frac{dw_1}{2\pi} w_1^2 X_i^a(w_1) X^{ib}(-w_1) \int \frac{dw}{2\pi} \frac{1}{(w^2 - R^2)} \frac{1}{[(w - w_1)^2 - R^2]} = 0 \quad (4.41)$$

which shows that the 2-point contribution to the effective action at one-loop is zero. We can also generalize this result to show that all possible non-derivative insertions on a loop with two derivatives will not give a contribution to the effective action.

4.2.3 V^4/R^7

Since all terms with two derivatives, no derivatives, or a mixture cancel by the arguments given above, the only possible non-vanishing term with four external fields is the four derivative term given by

$$-\frac{1}{4}Tr[(\Delta_F\Delta_J)^4] \quad (4.42)$$

or in frequency space

$$\int \frac{dw_2 dw_3 dw_4 dw - (w_2 + w_3 + w_4) X_i^8 [-(w_2 + w_3 + w_4)] w_2 X_j^8(w_2) w_3 X_k^8(w_3) w_4 X_l^8(w_4)}{(2\pi)^4 [(w + w_2)^2 - R^2] [(w + w_2 + w_3)^2 - R^2] [(w + w_2 + w_3 + w_4)^2 - R^2] [w^2 - R^2]} \quad (4.43)$$

with the prefactor

$$-\frac{1}{4} 2^4 Tr[(t^8)^4] Tr[(\mathcal{J}^{0i})^4] \quad (4.44)$$

for the gauge boson case. An identical result holds for fermions if one replaces the Lorentz generator trace with

$$Tr[S^{0i} S^{0j} S^{0k} S^{0l}] = 2(g^{ij} g^{kl} - g^{ik} g^{jl} + g^{il} g^{jk}) \quad (4.45)$$

whereas for the gauge bosons one finds

$$Tr[\mathcal{J}^{0i} \mathcal{J}^{0j} \mathcal{J}^{0k} \mathcal{J}^{0l}] = (g^{ij} g^{kl} + g^{il} g^{jk}). \quad (4.46)$$

Now using (4.21) and the low energy approximation $w_1, w_2, w_3, w_4 \rightarrow 0$, we get

$$-\frac{27i}{4} [(F_{0i}^8)^2]^2 \int \frac{dw}{2\pi} \frac{1}{(w^2 - R^2)^4}. \quad (4.47)$$

The integral can be performed in the complex plane using the usual $+i\epsilon$ prescription for handling the poles. Defining $(\dot{X}_i^8)^4 = (F_{0i}^8)^4 \equiv V^4$, one is left with the result that the first non-vanishing contribution to the effective potential has four derivatives,

$$\mathcal{L}_{eff}^{(1)} = \frac{27}{4} \frac{5}{32} \frac{V^4}{R^7}, \quad (4.48)$$

even when the gauge group experiences multiple levels of breaking.

4.2.4 $V^4 x^2 / R^9$ and $V^4 v^2 / R^9 r^5$

Looking at possible insertions with two background fields on a massive loop with four derivatives gives terms of the form,

$$Tr[(\Delta_F \Delta_J)^4 \Delta'_2] \quad (4.49)$$

$$-\frac{5}{2}Tr[(\Delta_F \Delta_J)^4 (\Delta_F \Delta_R)^2] \quad (4.50)$$

$$-\frac{5}{2}Tr[(\Delta_F \Delta_J)^4 (\Delta_F \Delta_r)^2] \quad (4.51)$$

$$-5Tr[(\Delta_F \Delta_J)^4 (\Delta_F \Delta_R)(\Delta_F \Delta_r)]. \quad (4.52)$$

The operators in (4.49) and (4.50) lead to terms of the form $V^4 x^2 / R^9$ with x being a light field (scale r) in agreement with [35], whereas the operators in (4.51) and (4.52) give terms with more powers of R in the denominator. At this point in our analysis, one might worry that we have thrown out the vertices coupling three quantum fields (two of mass R and one of mass r) with one background field which was found to be important in the result of [43]. However, by considering the x 's as background plus quantum fields, the effective operator $V^4 x'^2 / R^9$ contains the sum of all non-vanishing vertices with up to four derivatives constructable from such a vertex. We can now use $V^4 x'^2 / R^9$ in the path integral (4.20) and integrate over the light modes x' to generate

$$\frac{V^4 v^2}{R^9 r^5}, \quad (4.53)$$

where $v^2 \equiv (\dot{X}_i^3)^2$. Clearly (4.53) has the wrong dependence on R and r to reproduce the term of interest in the supergravity scattering amplitude.

4.3 Comment on the Eikonal approximation

When analyzing D0-brane scattering most authors (see e.g. [25, 43] and references therein) have chosen to use an explicit background given by $x = vt + b$ where v is a

relative velocity of the D0-branes and b an impact parameter. Such an approach allows one to construct the exact propagator as a power series in b , v , and t . By organizing the calculation along the lines suggested by our analysis above, we can exhibit the cancellation of all $V^4 v^2 / R^7 r^7$ contributions to the effective action. The point, again, is to take advantage of the large R limit. In the functional integral, one first does the integration over the fields with mass of order R . As explained in section 4.2.1 terms involving only D^2 cancel, allowing one to write a simplified expression for the effective action which only depends on the difference of the derivative terms between bosons and fermions

$$\Gamma[X] = \frac{i}{2} \text{Tr} \ln[1 + \Delta'_F \Delta_J^B] - \frac{i}{8} \text{Tr} \ln[1 + \Delta'_F \Delta_J^\psi], \quad (4.54)$$

where $\Delta'_F \equiv -D^{-2}$ is the propagator for the heavy fields and is a function of the background and the light fields. Again, terms with two derivatives of the background or light fields cancel as in (4.41). Terms with four derivatives and factors of r^2 expanded up from the heavy propagator yield precisely the structure $V^4 x^2 / R^9$. So again, there are no terms of the form $V^4 v^2 / R^7 r^7$ in the effective action.

This of course does not mean that there are not individual diagrams with the behavior $V^4 v^2 / R^7 r^7$. However, we see explicitly from this analysis that there are cancellations between bosons and fermions. In [43], a particular diagram with this behavior was exhibited. But we see that this contribution is cancelled by diagrams involving fermions.

Having seen by explicit calculation that the matrix model effective action contains no terms of the form $V^4 v^2 / R^7 r^7 \propto v_3^4 v_{12}^2 / R^7 r^7$, we show below in a calculation of the matrix model S -matrix how this term is extracted from the effective action.

4.4 Computing the S -Matrix in the Matrix Model

The matrix model is the dimensional reduction of ten dimensional supersymmetric Yang-Mills theory. The action is

$$S = \int dt \left[\frac{1}{g} \text{tr}(D_t X^i D_t X^i) + \frac{1}{2g} M^6 R_{11}^2 \text{tr}([X^i, X^j][X^i, X^j]) + \right. \quad (4.55)$$

$$\left. \frac{1}{g} \text{tr}(i\theta^T D_t \theta + M^3 R_{11} \theta^T \gamma^i [X^i, \theta]) \right]$$

where R_{11} is the eleven dimensional radius, M is the eleven dimensional Planck mass and $g = 2R_{11}$. The θ 's are the fermionic coordinates.

At small transverse velocity and small momentum transfer (with zero q^+ exchange) it is a straightforward matter to compute graviton-graviton scattering in the matrix model. One considers widely separated gravitons, and integrates out the high frequency modes of the matrix model. This yields, at one loop, an effective Lagrangian for the remaining diagonal degrees of freedom which behaves as

$$\mathcal{L}_{eff} = \frac{15}{16} \frac{v^4}{r^7} + \text{fermionic terms.} \quad (4.56)$$

If this effective Lagrangian is then treated in Born approximation, one reproduces precisely the supergravity result for the S -matrix.

Ref. [35] focused on the problem of multigraviton scattering in the matrix model. For three graviton scattering, it is necessary to compute the terms of order v^6 at two loops in the matrix model Hamiltonian. In the three graviton case, there are two relative coordinates and correspondingly two relative velocities. The basic strategy of [35], which will also be the strategy here, was to consider the case where one of the relative separations, say $x_{13} = x_1 - x_3 = R$, was much larger than $x_{12} = r$. In this limit, oscillators with frequency

of order R can be integrated out first, yielding an effective Lagrangian for those with mass (frequency) of order r (or zero). This effective Lagrangian is restricted by $SU(2)$ symmetry. Finally, one can consider integrating out oscillators with mass of order r .

In computing the S -matrix for three graviton scattering, as discussed already in [35], it is necessary not only to compute the terms of order v^6 in the effective action, but also to consider terms in the scattering amplitude which are of second order in the *one loop* (v^4) effective action. In other words, working with the effective action, it is necessary to go to higher order in the Born series.

In [35], it was observed that terms of the form

$$\frac{v_3^4 v_{12}^2}{R^7 r^7} \quad (4.57)$$

cannot appear in the *effective action* of the matrix model. As we have seen in sections 4.2 and 4.3, the v_3 factors can only arise from couplings to heavy fields. Integrating out the fields with mass of order R at one loop, the leading terms involving the light fields x^a are of the form $v_3^4 x^a x^a / R^9$. Moreover, it was argued that the terms in (4.57) were not generated by the higher order Born series referred to above. This last point, however, is incorrect, and is the source of the error. In fact, it is possible to find the corresponding term in the matrix model S -matrix.

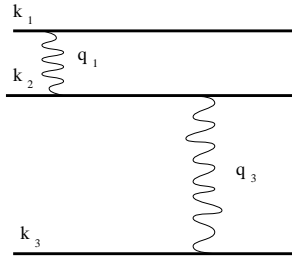


Figure 4.1: Ladder contribution to the supergravity amplitude. Solid lines are the scattering gravitons. Wiggly lines represent virtual gravitons with zero longitudinal momentum.

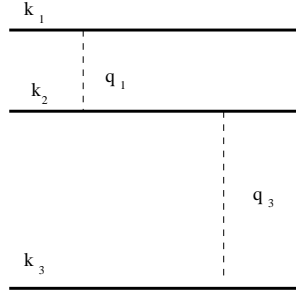


Figure 4.2: Corresponding contribution in the matrix model involving iteration of the one loop Hamiltonian. Dashed lines represent matrix elements of the interaction potential.

Consider the problem first from a Hamiltonian viewpoint. We wish to compare the supergravity graph of fig. 4.1 with the contribution of fig. 4.2 in old fashioned (time-ordered) perturbation theory. The second graph represents the iteration of the one loop effective Hamiltonian to second order. In momentum space, it has the correct $\frac{1}{q_3^2 q_1^2}$ behavior to reproduce the $\frac{1}{R^4 r^4}$ behavior of the missing supergravity S -matrix term. However, it has also an energy denominator, and various factors of velocity. It is straightforward to check that this energy denominator is proportional to $\frac{1}{2k_2 \cdot q_1 + q_1^2}$, the propagator appearing in the covariant diagram of fig. 4.1. To compare the diagrams in more detail, one also needs matrix elements of the type $\langle \vec{k}_i + \vec{q} | H' | \vec{k}_i \rangle$ where H' is the one loop Hamiltonian. As we will see in section 4.4.2, the leading term in powers of momentum transfer reproduces the corresponding term in the supergravity diagram. In other words, if one ignores the difference in the momenta of the particles in the initial, final and intermediate states, one obtains exact agreement. To see if higher order terms can cancel the energy denominator and reproduce the missing term, it is necessary to keep at least terms linear in the momentum transfer. The problem is that it is not clear how the momentum and r factors are to be ordered in the Hamiltonian. Depending on what one assumes about this ordering, one obtains quite different answers.

Of course, the full model has no such ordering problem. It is only our desire to simplify the calculation using the effective Hamiltonian that leads to this seeming ambiguity. There is an alternative approach, however, which leads to an unambiguous answer, and where one can exploit the simplicity of the one loop effective action. This is to use the path integral. As we will see, the path integral approach permits an unambiguous resolution of the ordering problem.

4.4.1 Path integral Computation of the S-Matrix

Let us consider the problem of computing the S -matrix using the path integral. We will use an approach which is quite close to the eikonal approximation (it is appropriate for small angle scattering) which has been used in most analyses of matrix model scattering. It is helpful, first, to review some aspects of potential scattering. In particular, let us first see how to recover the Born approximation by studying motion near a classical trajectory.

A useful starting point is provided in [37]. In the path integral, it is most natural to compute the quantity

$$\langle \vec{x}_f | e^{-iHT} | \vec{x}_i \rangle = \int [dx] e^{iS}. \quad (4.58)$$

To compute the S -matrix, one wants to take the initial and final states to be plane waves, so one multiplies by $e^{i\vec{p}_i \cdot \vec{x}_i} e^{-i\vec{p}_f \cdot \vec{x}_f}$ and integrates over x_i and x_f . For small angle scattering in a weak, short-ranged potential, one expects that the dominant trajectories are those for free particles,

$$\vec{x}_o(t) = \frac{\vec{x}_i + \vec{x}_f}{2} + \vec{v}t \quad (4.59)$$

where t runs from $-\frac{T}{2}$ to $\frac{T}{2}$, and $\vec{v} = \frac{\vec{x}_f - \vec{x}_i}{T}$. It is convenient to change variables [37] to \vec{v}

and \vec{b} ,

$$\vec{b} = \frac{\vec{x}_f + \vec{x}_i}{2}. \quad (4.60)$$

The complete expression for the amplitude is then

$$\mathcal{A}_{i \rightarrow f} = \int d^9 v \int d^9 b e^{i\vec{b} \cdot (\vec{p}_f - \vec{p}_i)} e^{i\vec{v} \cdot (\vec{p}_i + \vec{p}_f)} \int [d\vec{x}] e^{iS}. \quad (4.61)$$

Now if we expand the classical action about this solution, writing

$$\vec{x} = \vec{x}_o + \delta\vec{x}, \quad (4.62)$$

(note $\delta\vec{x}$ includes both classical corrections to the straight line path and quantum parts) we have a free piece,

$$S_o = v^2 T / 2. \quad (4.63)$$

For large T , the v integral can be done by stationary phase, yielding

$$\vec{v} = \frac{\vec{p}_f + \vec{p}_i}{2}. \quad (4.64)$$

We will see that this effectively provides the ordering prescription we require for the matrix model problem.

For the case of potential scattering, expand e^{iS} in powers of V , and replace the potential by its Fourier transform. The leading semiclassical contribution to the amplitude is then proportional to

$$\int d^9 b e^{i\vec{b} \cdot (\vec{p}_f - \vec{p}_i)} \int d^9 q \int dt V(q) e^{i\vec{q} \cdot (\vec{b} + \vec{v}t)}. \quad (4.65)$$

The t integral gives a δ -function for energy conservation, while the b integral sets $\vec{q} = \vec{p}_f - \vec{p}_i$.

This is precisely the Born approximation result.

Higher terms in the Born series can be worked out in a similar fashion. Time ordering the terms and replacing the potential by its Fourier transform, the time integrals *almost* give the expected energy denominators. The terms linear in momentum transfer (involving $\vec{v} \cdot \vec{q}$) are given correctly, but the \vec{q}^2 terms are not. These terms must be generated by the expansion of V in powers of δx , which generates additional powers of \vec{q} . This problem, which is essentially the problem of recoil discussed in [44], will be analyzed in a separate publication [36]. Here we will work to leading order in q , and second order in V .

At second order in V , we need to consider an expression of the form

$$\int dv \int db e^{i\vec{v} \cdot \frac{\vec{p}_i + \vec{p}_f}{2}} e^{i\vec{b} \cdot (\vec{p}_f - \vec{p}_i)} e^{-\frac{v^2 T}{2}} \int_{-\frac{T}{2}}^{\frac{T}{2}} dt_1 \int_{-\frac{T}{2}}^{\frac{T}{2}} dt_2 V(\vec{x}(t_1)) V(\vec{x}(t_2)). \quad (4.66)$$

Time order the t_1, t_2 integrals, and Fourier transform each of the factors of V . The integral over \vec{v} is again done by stationary phase, and the resulting expression has the form:

$$\frac{1}{2!} \int_{-\frac{T}{2}}^{\frac{T}{2}} dt_1 \int_{-\frac{T}{2}}^{t_1} dt_2 \int db \int dq_1 \int dq_2 \mathcal{V}(\vec{q}_1) \mathcal{V}(\vec{q}_2) e^{i \cdot b(\vec{p}_f - \vec{p}_i) + i\vec{q}_1 \cdot (\vec{b} + \vec{v}t_1) + i\vec{q}_2 \cdot (\vec{b} + \vec{v}t_2)} \quad (4.67)$$

It is now straightforward to do the t_i , \vec{q}_i , and \vec{b} integrals. The integral over t_2 yields the energy denominator, $\frac{1}{\vec{v} \cdot \vec{q}_2}$. This differs from the exact energy denominator by terms of order q^2 . The final integral over t_1 yields the overall energy conserving δ -function. Up to these terms of order q^2 , this is exactly the second order Born approximation expression.

4.4.2 The Ladder Graphs

We are now in a position to compare the supergravity and matrix model ladder graphs (see fig. 4.1 and 4.2). On the supergravity side, the calculation is completely standard, and proceeds along the lines of [35]. As there, we take the vertices from [29]² and

²To be consistent with the authors [30, 31] who use $\kappa^2 = 16\pi^5$, the 3-vertex in [29] needs to be multiplied by 2.

require that the polarizations of the incoming and outgoing gravitons be identical (as is true to leading order in the inverse distance in the matrix model). The $N_2 N_3 \frac{v_3^4}{q_3^2}$ term comes from the second vertex, and is precisely of the same form as in graviton-graviton scattering. The vertex on the first graviton line is

$$-k_{1\sigma}(k_{1\gamma} - q_{1\gamma}) - (k_{1\sigma} - q_{1\sigma})k_{1\gamma} + 2k_{1\sigma}k_{1\gamma} + 2(k_{1\sigma} - q_{1\sigma})(k_{1\gamma} - q_{1\gamma}). \quad (4.68)$$

From the first vertex on the second graviton line, we get a similar expression, replacing k_1 with k_2 and q_1 by $-q_1$. Multiplying these factors together, and including the propagator, gives for the corresponding amplitude:

$$\mathcal{A}_1 = (2\kappa)^4 (k_2 \cdot k_3)^2 (k_1 \cdot k_2) \frac{[(k_1 \cdot k_2) - (q_1 \cdot k_2) + \mathcal{O}(q_1^2)]}{q_1^2 q_3^2 (2k_2 \cdot q_1 + q_1^2)}. \quad (4.69)$$

Or

$$\mathcal{A}_1 = \frac{\kappa^4 N_1 N_2^2 N_3 v_3^4 v_{12}^2 [(v_{12}^2 - \frac{2}{N_1} \vec{q}_1 \cdot \vec{v}_{12}) + \mathcal{O}(q_1^2)]}{16 q_1^2 q_3^2 (\vec{q}_1 \cdot \vec{v}_{12})}, \quad (4.70)$$

expressed in light cone variables with non-relativistic normalization.

Now we want to compare with the matrix model prediction, fig. 4.2. Recalling the averaging prescription, for the matrix elements of the interaction Hamiltonian we have (dropping terms suppressed by extra powers of q_3)

$$\left(\frac{15}{16}\right)^2 N_1 N_2^2 N_3 v_3^4 \left(\frac{\vec{k}_1}{N_1} - \left(\frac{\vec{k}_2}{N_2}\right) - \frac{1}{2} \vec{q}_1 \left(\frac{1}{N_1} + \frac{1}{N_2}\right) \right)^4 \quad (4.71)$$

$$= \left(\frac{15}{16}\right)^2 N_1 N_2^2 N_3 v_3^4 v_{12}^2 [(v_{12}^2 - \frac{2}{N_1} \vec{q}_1 \cdot \vec{v}_{12}) - \frac{2}{N_2} (\vec{q}_1 \cdot \vec{v}_{12}) + \mathcal{O}(q_1^2)]. \quad (4.72)$$

After Fourier transforming r and R (not shown above), the first term is exactly the term found on the supergravity side. The second term cancels the energy denominator, yielding

a contact term,

$$-\kappa^4 N_1 N_2 N_3 \frac{v_3^4}{8} \frac{v_{12}^2}{q_1^2 q_3^2}. \quad (4.73)$$

Each of the four ladder diagrams yields an identical contribution. The sum is precisely the “missing” term of [35]. At this level, there is no discrepancy between the DLCQ prediction for the scattering amplitude and supergravity.

4.5 Additional contributions to three graviton scattering

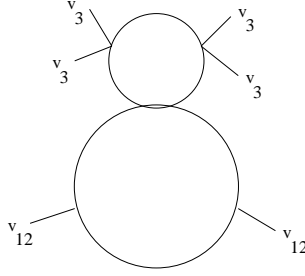


Figure 4.3: Matrix model contribution to three graviton amplitude.

In section 4.2 we showed how to obtain terms in the matrix model effective action with four powers of $v_3 \propto V$ and two powers of $v_{12} = v$ (see fig. 4.3). In this section we derive the coefficient of (4.53) using other techniques. Recall that integrating out fields with mass of order R yields no terms independent of velocities or quadratic in velocities; at quartic order in velocity, one has:

$$\mathcal{L} = \frac{15}{16} \left(\frac{v_{13}^4}{|\vec{x}_{13}|^7} + \frac{v_{23}^4}{|\vec{x}_{23}|^7} \right). \quad (4.74)$$

For small x_{12} , one can expand in powers of x_{12} . The result can be generalized to an $SU(2)$ invariant expression:

$$\delta\mathcal{L} = \frac{15}{64} v_3^4 [((\vec{x}_1 + \vec{x}_2) \cdot \nabla_R)^2 + (\vec{x}^a \cdot \nabla_R)^2] \frac{1}{R^7}. \quad (4.75)$$

Here $x_1 + x_2$ is the center of mass of the 1 – 2 system (combined with the leading term, the expression is translationally invariant). The superscript a is an $SU(2)$ index. Contracting the x^a factors, the leading (infrared divergent and finite) terms cancels as we saw in the previous chapter. The Euclidean propagator, up to terms quadratic in velocities is given by

$$\langle x^{+i} x^{-j} \rangle = \frac{\delta^{ij}}{\omega^2 + r^2} + \frac{4(v^i v^j) + \text{const } \delta^{ij} v^2}{(\omega^2 + r^2)^3}. \quad (4.76)$$

Substituting back in our expression above and performing the frequency integral yields

$$N_1 N_2 N_3 \frac{45}{64 R_{11}^5} \frac{v_3^4}{r^5} (\vec{v}_{12} \cdot \nabla)^2 \frac{1}{R^7}. \quad (4.77)$$

In deriving this expression it is necessary to keep track of various factors of 2. One comes from the two real massive fields in the loop (or equivalently, written in terms of complex fields, from an extra 2 which appears in the vertex), the other from a factor of $g = 2R_{11}$ for a 2-loop result. It is easy to show that this is the only contribution with this r dependence and four factors of v_3 .

Let's compare this with the supergravity amplitude. There is only one diagram with the tensor structure of (4.77); this arises from the diagram of fig. 4.4. There are also several terms in individual diagrams of the form $v_3^4 v_{12}^2 \frac{1}{R^8 r^6}$, as well as terms of order $1/R^9 r^5$ with a different tensor structure than the matrix model result. We will shortly explain that, at the level of the S -matrix, all of these terms match, just as in the case of the leading $1/R^7 r^7$ term.

Let us first consider the contribution to the supergravity S -matrix of the form (4.77) above. The relevant diagram is shown fig. 4.4. It is convenient to view q_1 and q_3 as independent, so $q_2 = -q_1 - q_3$. From [29], the necessary piece of the three graviton vertex

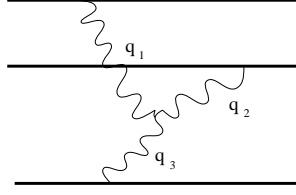


Figure 4.4: Contribution to three graviton amplitude involving three graviton vertex.

is

$$2P_3(k_{1\sigma}k_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + 2P_6(k_{1\sigma}k_{1\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + 4P_6(k_{1\nu}k_{2\gamma}\eta_{\beta\mu}\eta_{\alpha\sigma}) + 4P_3(k_{1\nu}k_{2\mu}\eta_{\beta\sigma}\eta_{\gamma\alpha}). \quad (4.78)$$

It is then a straightforward exercise to evaluate the diagram. Matters are considerably simplified by using kinematic relations such as $k_1 \cdot q_1 = -\frac{1}{2}q_1^2$, $k_1 \cdot q_2 = \frac{1}{2}q_1^2 - k_1 \cdot q_3$, etc., and dropping terms with the wrong R dependence. After only a few lines of algebra, this yields the covariant form of the amplitude:

$$\frac{16\kappa^4[(k_1 \cdot k_3)^2(k_2 \cdot q_3)^2 + (k_2 \cdot k_3)^2(k_1 \cdot q_3)^2 - 2(k_1 \cdot k_3)(k_2 \cdot k_3)(k_1 \cdot q_3)(k_2 \cdot q_3)]}{q_1^4 q_3^2}. \quad (4.79)$$

Changing to light cone variables with non-relativistic normalization gives

$$\frac{\kappa^4 N_1 N_2 N_3}{2R_{11}^3} v_3^4 ((\vec{v}_1 - \vec{v}_2) \cdot \vec{q}_3)^2 \frac{1}{q_1^4 q_3^2}. \quad (4.80)$$

Then Fourier transforming gives precisely the matrix model result (4.77).

There are several other kinematic structures which appear in individual super-gravity diagrams which do not arise in the matrix model computation, and thus must be produced by iteration of the one loop action. The cancellation, in fact, is closely related to the cancellation we have studied of the leading term. For example, there are terms from the diagram of fig. 4.4 which behave as $\frac{v_3^4 v_{12}^2}{R^8 r^6}$. To see how this and other terms cancel, let us return to our earlier discussion. There, we set $q_1 = -q_2$. However, we should be more

careful, and write $q_2 = -q_1 - q_3$. Then from fig. 4.4 we have a contribution

$$-\frac{\kappa^4}{2}N_1N_2N_3v_3^4v_{12}^2\left(\frac{q_1 \cdot q_2 + q_1 \cdot q_3 + q_2 \cdot q_3}{q_1^2q_2^2q_3^2}\right) \quad (4.81)$$

(previously we kept only the first term and set $q_1 = -q_2$). We also have the supergravity term involving the 4-vertex discussed in [35]

$$-\frac{\kappa^4}{2}N_1N_2N_3v_3^4v_{12}^2\left(\frac{1}{q_1^2q_3^2} + \frac{1}{q_2^2q_3^2}\right). \quad (4.82)$$

On the matrix model side, the higher order Born terms yield

$$-\frac{\kappa^4}{4}N_1N_2N_3v_3^4v_{12}^2\left(\frac{1}{q_1^2q_3^2} + \frac{1}{q_2^2q_3^2}\right). \quad (4.83)$$

As before, the leading terms match. Expanding in powers of q_3 , it is not hard to check that the coefficients of $q_1 \cdot q_3$ and $(q_1 \cdot q_3)^2$ match as well.

4.6 More Gravitons

4.6.1 n-Graviton Scattering

Certain terms in the four and higher graviton scattering amplitude are easily evaluated by these methods. On the matrix model side, the calculations are particularly simple. One can, for example, consider a generalization of the three graviton calculation above, indicated in fig. 4.5. At two loops, we saw that we generate in $SU(3)$ an effective coupling,

$$\frac{45}{64}v_3^4(\vec{v}_{12} \cdot \vec{\nabla}_R)^2 \frac{1}{R^7 r^5}. \quad (4.84)$$

We can generalize this to the case of $SU(4)$, with the hierarchy $x_{4i} \gg x_{3\ell} \gg x_{21}$, where $i = 1, 2, 3$ and $\ell = 1, 2$. In other words, we again suppose that there is a hierarchy of distance scales, with one particle very far from the other three, and one of these three far

from the remaining two. Again, we proceed by first integrating out the most massive states, then the next most massive, and so on. After the first two integrations, we generate a term (among others)

$$\frac{45}{64} v_4^4 (\vec{v}_3 \cdot \nabla_4)^2 \frac{1}{|\vec{x}_4|^7} \left(\frac{1}{|\vec{x}_{31}|^5} + \frac{1}{|\vec{x}_{32}|^5} \right). \quad (4.85)$$

As before, expand this term in powers of the small distances x_1 , x_2 , and generalize to an $SU(2)$ -invariant expression, yielding:

$$\frac{45}{256} v_4^4 (\vec{v}_3 \cdot \nabla_4)^2 \frac{1}{|\vec{x}_4|^7} (\vec{x}^a \cdot \nabla_3)^2 \frac{1}{|\vec{x}_3|^5}. \quad (4.86)$$

Finally, the integration of x_a yields various terms. The piece of $\langle x_a^i x_a^j \rangle \propto v^i v^j$ gives

$$\frac{135}{256} v_4^4 (\vec{v}_3 \cdot \vec{\nabla}_4)^2 \frac{1}{|\vec{x}_4|^7} (\vec{v}_{12} \cdot \vec{\nabla}_3)^2 \frac{1}{|\vec{x}_3|^5} \frac{1}{|\vec{x}_{12}|^5}. \quad (4.87)$$

Higher order terms corresponding to n -graviton scattering generated in a similar fashion will be discussed below.

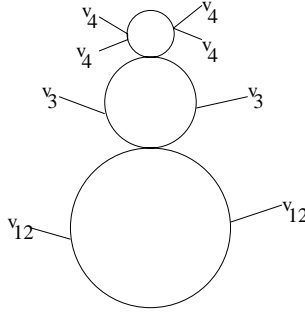


Figure 4.5: A matrix model diagram contributing to four graviton scattering.

Another term which is easily obtained is indicated in the diagram in fig. 4.6. This graph includes the interaction of the light $SU(2)$ fields from integrating out the fields with mass of order x_4 at one loop, as well as those obtained by integrating out the fields of mass

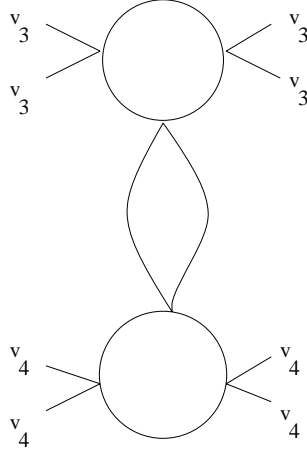


Figure 4.6: Another diagram which is easily computed.

x_3 at one loop. The relevant interactions are

$$\left(\frac{15}{64}\right)^2 (v_4^4 v_3^4) [(\vec{x}^a \cdot \vec{\nabla}_4)^2 \frac{1}{|\vec{x}_4|^7} (\vec{x}^a \cdot \vec{\nabla}_3)^2 \frac{1}{|\vec{x}_3|^7}] \quad (4.88)$$

Now contracting the x^a factors as in fig. 4.6 yields a term:

$$4\left(\frac{15}{64}\right)^2 \left[v_4^4 v_3^4 (\nabla_4^i \nabla_4^j) \frac{1}{|\vec{x}_4|^7} (\nabla_3^i \nabla_3^j) \frac{1}{|\vec{x}_3|^7} \right] \frac{1}{|\vec{x}_{12}|^3} \quad (4.89)$$

On the supergravity side, the required computations are somewhat more complicated. The easiest to consider is the first term (4.87) above. This term is generated by the diagram of fig. 4.7. It is not difficult to find the particular tensor structures which give the matrix model expression (4.87). Focus first on the terms involving $\vec{v}_3 \cdot \vec{\nabla}_4$. These must come from dotting k_3 into q'_2 or q_4 . Calling $q'_2 = -q_3 - q_4$, the relevant term in the three graviton vertex is (μ, α are the polarization indices carried by the graviton with momentum q'_2)

$$2 \left[P_3(q'_{2\sigma} q_{3\gamma} \eta_{\mu\nu} \eta_{\alpha\beta}) + P_6(q'_{2\sigma} q'_{2\gamma} \eta_{\mu\nu} \eta_{\alpha\beta}) + 2P_6(q'_{2\nu} q_{3\gamma} \eta_{\beta\mu} \eta_{\alpha\sigma}) + 2P_3(q'_{2\nu} q_{3\mu} \eta_{\beta\sigma} \eta_{\gamma\alpha}) \right] \quad (4.90)$$

Only a few permutations actually contribute, and contracting with the scattering graviton

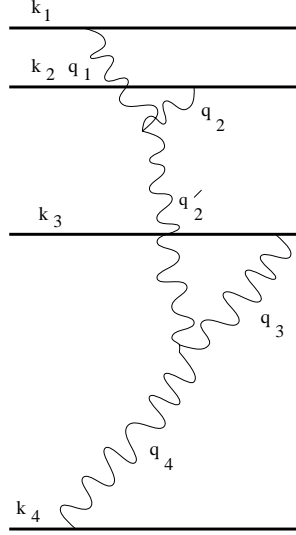


Figure 4.7: A supergravity contribution to four graviton scattering.

momentum, $k_{4\sigma}k_{4\gamma}k_{3\nu}k_{3\beta}$, gives

$$2(k_3 \cdot q_4)^2 k_{4\mu} k_{4\alpha} \quad (4.91)$$

So the whole diagram collapses to $\frac{2(k_3 \cdot q_4)^2}{q_3^4}$ times the three graviton term we evaluated earlier. The result agrees completely with the matrix model computation (4.87).

Indeed, one can now go on to consider similar terms in n -graviton scattering. The supergravity graph indicated in fig. 4.8 can be evaluated by iteration. The coupling of the $n - 1$ graviton is similar to that of the third graviton in the 4-graviton amplitude and can be treated in an identical fashion. The result then reduces to the $n - 1$ graviton computation. So one obtains

$$\frac{15}{16} \left(\frac{3}{4}\right)^{n-2} v_n^4 (\vec{v}_{n-1} \cdot \vec{\nabla}_n)^2 \frac{1}{|\vec{x}_n|^7} (\vec{v}_{n-2} \cdot \vec{\nabla}_{n-1})^2 \frac{1}{|\vec{x}_{n-1}|^5} \dots (\vec{v}_3 \cdot \vec{\nabla}_4)^2 \frac{1}{|\vec{x}_4|^5} (\vec{v}_{12} \cdot \vec{\nabla}_3)^2 \frac{1}{|\vec{x}_3|^5} \frac{1}{|\vec{x}_{12}|^5} \quad (4.92)$$

The corresponding term in the matrix model effective action is also obtained by iteration. It is easy to generalize the calculation of fig. 4.5 to the case above. Repeating

our earlier computations gives precisely the result of eqn. (4.92) above.

The computation of the part of the supergravity amplitude corresponding to eqn. (4.89) is more complicated. This term is generated by the sum of several diagrams. We will not attempt a detailed comparison here, leaving this, as well as certain other terms, for future work.

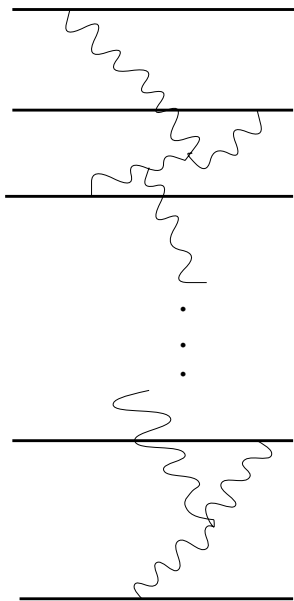


Figure 4.8: Diagram contributing to n graviton scattering.

4.6.2 Other Dimensions

According to the Matrix model hypothesis, the compactification of M -theory to $11 - k$ dimensions is described by $k + 1$ dimensional super Yang-Mills theory [45, 46]. For graviton-graviton scattering, this has been done in [24]. It is a simple matter to extend our analysis to these cases.

As an illustration, consider the three graviton case. Working in units where the compact dimensions have $R_k = 1$, then the Fourier transforms needed to convert the super-

gravity result (4.80) to an effective potential are

$$\frac{\kappa^2 v_3^4}{4(2\pi)^{1+k}} \int \frac{d^{9-k} q_3}{(2\pi)^{9-k}} \frac{e^{i\vec{q}_3 \cdot \vec{R}}}{q_3^2} = \frac{v_3^4}{2^{1+k}(\sqrt{\pi})^{1+k}} \frac{1}{R^{7-k}} \Gamma\left(\frac{7-k}{2}\right) \quad (4.93)$$

and

$$\frac{2\kappa^2 v_{12}^i v_{12}^j}{(2\pi)^{1+k}} \int \frac{d^{9-k} q_1}{(2\pi)^{9-k}} \frac{e^{i\vec{q}_1 \cdot \vec{r}}}{q_1^4} = \frac{v_{12}^i v_{12}^j}{2^k(\sqrt{\pi})^{1+k}} \frac{1}{r^{5-k}} \Gamma\left(\frac{5-k}{2}\right). \quad (4.94)$$

On the matrix model side, the loop integrals arising from integrating out the massive states must now be performed in $k+1$ dimensions giving

$$-6v_3^4 \int \frac{d^{1+k} p}{(2\pi)^{1+k}} \frac{1}{(p^2 + R^2)^4} = \frac{v_3^4}{2^{1+k}(\sqrt{\pi})^{1+k}} \frac{1}{R^{7-k}} \Gamma\left(\frac{7-k}{2}\right) \quad (4.95)$$

and

$$4v_{12}^i v_{12}^j \int \frac{d^{1+k} p}{(2\pi)^{1+k}} \frac{1}{(p^2 + r^2)^3} = \frac{v_{12}^i v_{12}^j}{2^k(\sqrt{\pi})^{1+k}} \frac{1}{r^{5-k}} \Gamma\left(\frac{5-k}{2}\right) \quad (4.96)$$

in agreement with the supergravity result above. All of the integrals are convergent for $k \leq 4$ and since these same integrals are needed for our n -graviton result, it is a simple matter to show that the agreement we have found here persists for arbitrary n .

4.7 Some Puzzles

In the original discussion of [10], as well as in [16], the question was raised: why does the lowest order matrix model calculation reproduce the tree level supergravity result for graviton-graviton scattering. The scattering amplitude is given by a power series in $\frac{gN}{r^3}$, and one ultimately wants to take a limit with $N \rightarrow \infty$, g fixed. Moreover, one wants to take this limit uniformly in r , i.e. one does not expect to scale distances with N . The answer suggested by these authors was that the explanation lies in a non-renormalization theorem

for v^4 terms, which insures that the one-loop result is exact. The required cancellation was demonstrated at two loops in [25]. Such a theorem for four derivative terms in four dimensional field theory was proven in [33]. The complete proof for the matrix model was finally provided in [32].

The agreement of three graviton scattering in the matrix model with supergravity suggests that there are more non-renormalization theorems governing the various possible terms at order v^6 . Indeed, a proof was provided for $SU(2)$ in [47] and for $SU(3)$ in [48]. On the other hand, it is rather easy to see, following reasoning similar to that of [42], that there are operators at order v^6 which are renormalized in $SU(N)$, $N \geq 4$. In particular, consider the case of four gravitons. In the previous section, we computed the contribution to the amplitude (4.87) by contracting $x^a x^a$ in eqn. (4.76), and took the piece quadratic in v^2 . Taking, instead, the leading, velocity-independent term in this propagator yields a contribution to the effective action,

$$N_1 N_2 N_3 N_4 \frac{45}{256} v_4^4 (\vec{v}_3 \cdot \vec{\nabla}_4)^2 \frac{1}{x_4^7} \nabla_3^2 \frac{1}{x_3^5} \frac{1}{x_{12}}. \quad (4.97)$$

Not only does this represent a renormalization of the v^6 terms computed at two loops, but the N -dependence of (4.97) is not appropriate to a Lorentz-invariant amplitude. One might wonder if this term can be cancelled by terms generated at higher order in the Born series. However, to see that this is not the case, one can define an index of an amplitude, \mathcal{A} (written in momentum space), $I_{\mathcal{A}}$, which is simply the difference of the number of powers of momentum in the numerator and in the denominator. All of the amplitudes we have studied previously have $I_{\mathcal{A}} = 2$. The iterations of the lower order matrix Hamiltonian also have $I_{\mathcal{A}} = 2$. However, (4.97) has $I_{\mathcal{A}} = -4$. So this can not be the source of the discrepancy. We have checked carefully for other diagrams in the matrix model effective action which might

have this structure, and we do not believe there are any. It is interesting to note that the work of [48] also indicates that there are v^6 terms that are not necessarily protected from renormalizations for $N > 3$.

Chapter 5

Non-Renormalization Theorems for v^{2N} terms in $SU(N)$

Agreement between the matrix model and supergravity displayed in the previous chapter suggests an infinite sequence of non-renormalization theorems for a particular class of v^{2N} terms in the $SU(N)$ matrix model effective action for arbitrary N . The present chapter will investigate generalizations of the techniques developed in proving non-renormalization theorems for v^4 terms in $SU(N)$ and v^6 terms in $SU(3)$ [32, 47, 48, 49]. The basic strategy is to look at the supersymmetric completion of the v^{2N} terms involving the most fermions, θ^{4N} . By analyzing the differential equations resulting from a supersymmetric variation of the θ_{4N} terms, one hopes to fully constrain the tensor structure and scale (x_{ij}) dependence. We will find that without additional assumptions about the absence of acceleration terms in the matrix model effective action, we are unable to make definite statements for θ^{4N} terms in $SU(N)$. However, by investigating the implications of the v^6 result in $SU(3)$ [48], we can outline a proof showing that some of the v^{2N} terms from

chapter four are not renormalized.

5.1 Non-renormalization of eight fermion terms in $SU(2)$

It will be useful to review the arguments used in [32] to reach the conclusion that the supersymmetric completion [50, 51] of the v^4 term involving eight fermions are not renormalized for the $SU(2)$ matrix model effective action. We will see in section 5.2 and 5.3 that similar arguments can be applied to twelve fermion terms in $SU(3)$ with varying degrees of success.

The various terms in the supersymmetric completion of v^4 have been calculated by a number of different authors and are written together in [37] where they show complete agreement with the tree-level supergravity amplitude. The supersymmetric completion of v^4 includes terms with at most eight fermions, θ^8 , and no derivatives. The variation of θ^8 cannot be cancelled by other terms in the action and must therefore vanish. To see this, consider the free matrix model action for the massless modes in $SU(2)$,

$$\mathcal{L}_0 = \frac{1}{g} \int dt \left(\frac{1}{2} v^2 + i \theta \dot{\theta} \right) \quad (5.1)$$

and the one-loop supersymmetric completion for v^4/r^7 at order four¹,

$$\begin{aligned} \mathcal{L}_1 = \int dt \frac{15}{16} [v^4 + \frac{i}{2} v^2 v_m (\theta \gamma^{mn} \theta) \partial_n - \frac{1}{8} v_p v_q (\theta \gamma^{pm} \theta) (\theta \gamma^{qn} \theta) \partial_m \partial_n - \\ \frac{i}{144} v_q (\theta \gamma^{qm} \theta) (\theta \gamma^{nk} \theta) (\theta \gamma^{pk} \theta) \partial_m \partial_n \partial_p + \frac{1}{8064} (\theta \gamma^{ml} \theta) (\theta \gamma^{nl} \theta) (\theta \gamma^{pk} \theta) (\theta \gamma^{qk} \theta) \partial_m \partial_n \partial_p \partial_q] \frac{1}{r^7}. \end{aligned} \quad (5.2)$$

The tree-level susy transformation laws are

$$\begin{aligned} \delta x^i &= -i \epsilon \gamma^i \theta \\ \delta \theta &= \gamma^i v^i \epsilon, \end{aligned} \quad (5.3)$$

¹Terms of a given order (number of derivative plus one-half the number of fermions) are preserved by the tree-level susy transformations [50].

where the susy transformation parameter ϵ is order $-1/2$ and γ^i are the real symmetric Dirac matrices representing the $SO(9)$ Clifford algebra. However, in order for the supersymmetry algebra to close on shell for $\mathcal{L}_0 + \mathcal{L}_1$, the transformation laws must be corrected to

$$\begin{aligned}\delta x^i &= -i\epsilon\gamma^i\theta + \epsilon N^i\theta \\ \delta\theta &= \gamma^i v^i \epsilon + M\epsilon\end{aligned}\tag{5.4}$$

with N order 2 and M order 3 [32, 51]. Now it becomes apparent that the variation of the θ^8 term into a θ^9 term must be invariant since the variation of the tree-level fermionic kinetic term with M containing six fermions produces at most a seven fermion term.

Although we have written down the precise structure of the eight fermion term in (5.2) as first computed by [52], it can also be determined by writing down all the possible eight fermion tensor structures compatible with the $SO(9)$ symmetry and demanding invariance under susy transformations. As demonstrated in [32], the most general eight fermion terms compatible with $SO(9)$ invariance and CPT have the form,

$$(\theta\gamma^{ij}\theta\theta\gamma^{jk}\theta\theta\gamma^{lm}\theta\theta\gamma^{mn}\theta)(g_1(r)\delta_{in}\delta_{kl} + g_2(r)\delta_{kl}x_ix_n + g_3(r)x_ix_kx_lx_n).\tag{5.5}$$

Keeping only the nine fermion term after varying the bosonic coordinate of the functions $g_i(r)$ in (5.5) gives,

$$-i\gamma_{ab}^s\theta_b(\theta\gamma^{ij}\theta\theta\gamma^{jk}\theta\theta\gamma^{lm}\theta\theta\gamma^{mn}\theta)\partial_s(g_1(r)\delta_{in}\delta_{kl} + g_2(r)\delta_{kl}x_ix_n + g_3(r)x_ix_kx_lx_n).\tag{5.6}$$

For invariance of the Lagrangian to susy transformations, (5.6) must equal zero. One can proceed to determine the functions $g_i(r)$ by applying the operator $\gamma_{ac}^q \frac{d}{d\theta_c} \partial_q$ to (5.6) giving three coupled second order differential equations. By requiring that the solutions go to zero as $r \rightarrow \infty$, and do not contain negative or fractional powers of the coupling, one is left with

the eight fermion term written in the compact $1/r^7$ form shown in (5.2). Alternatively, one can apply the operator $\gamma_{ac}^q \frac{d}{d\theta_c} x_q$ to (5.6) to get three coupled first order differential equations with the only solutions being given by the eight fermion term in (5.2). Thus we see that the uniquely determined structure for the eight fermion term has the r dependence of a one-loop exact result. Consistency with the supersymmetry transformations implies that the supersymmetric completion of the θ^8 term, including v^4 also be non-renormalized [51] in agreement with our results in chapter three.

5.2 Supersymmetric completion of $\frac{v_3^4}{r^5}(\vec{v}_{12} \cdot \nabla_R)^2 \frac{1}{R^7}$

In chapter three and again in chapter four we generated an operator of the form,

$$\frac{15}{64} v_3^4 (\vec{x}^a \cdot \nabla_R)^2 \frac{1}{R^7}, \quad (5.7)$$

in $SU(3)$ by integrating out the most massive modes of scale R . Then by integrating out the lighter $SU(2)$ modes, x^a , of scale r , we arrive at the the matrix model term,

$$\frac{45}{64} \frac{v_3^4}{r^5} (\vec{v}_{12} \cdot \nabla_R)^2 \frac{1}{R^7}, \quad (5.8)$$

which was shown to agree with the supergravity amplitude for three-graviton scattering in the limit that the third graviton is far away from the remaining two. Based on the results of the previous section, if we want to prove an expected non-renormalization theorem for these v^6 terms in $SU(3)$, a natural place to begin is by studying the supersymmetric completion of (5.8) with the most fermions. However, unlike in the previous section, we will see below that there are two twelve fermion terms in the supersymmetric completion of (5.8) which complicates the analysis of proving a non-renormalization theorem.

To consider the supersymmetric completion of (5.8) with the most (twelve) fermions, we begin by following the steps used in section 4.5 to derive (5.7) for the analogous eight fermion term. Up to an overall constant the eight fermion term in $SU(2)$ has been determined by considering constraints imposed by supersymmetry in the previous section while an explicit background field calculation [52] gives the numerical coefficient, allowing us to write down the eight fermion term² in $SU(3)$ after integrating out the modes of scale R ,

$$\mathcal{L} = \frac{15}{16} \frac{1}{8064} \left(\frac{\theta_{13}^8}{|\vec{x}_{13}|^{11}} + \frac{\theta_{23}^8}{|\vec{x}_{23}|^{11}} \right). \quad (5.9)$$

As in section 4.5, expand in powers of x_{12} and generalize to an $SU(2)$ effective operator

$$\delta\mathcal{L} = \frac{15}{64} \frac{1}{8064} [\theta_3^8 (\vec{x}^a \cdot \nabla_R)^2] \frac{1}{R^{11}}. \quad (5.10)$$

Now using the background fermion Feynman rules developed in [52], it is a straightforward matter to contract the x^a 's to make a two loop twelve fermion term,

$$\begin{aligned} & \frac{15}{64} \frac{1}{8064} \theta_3^8 \left(\frac{10}{2^8} \right) \left[\frac{3}{r^7} (\theta_{12} \gamma^i \gamma^n \theta_{12}) (\theta_{12} \gamma^n \gamma^j \theta_{12}) \nabla_R^i \nabla_R^j - \right. \\ & \left. \frac{21}{r^9} (\theta_{12} \gamma^i \gamma^n \theta_{12}) (\theta_{12} \gamma^m \gamma^j \theta_{12}) r^n r^m \nabla_R^i \nabla_R^j - \frac{7}{r^9} (r \cdot \nabla_R) (\theta_{12} \gamma^i \gamma^n \theta_{12}) (\theta_{12} \gamma^n \gamma^j \theta_{12}) r^i \nabla_R^j \right] \frac{1}{R^{11}}. \end{aligned} \quad (5.11)$$

Schematically, each of the terms in (5.11) have the form $\theta_3^8 (\theta_{12}^{ij})^4 \nabla_R^i \nabla_R^j / R^{11} r^7$. A susy variation of such a term with respect to the bosonic coordinate R is invariant based on the argument given in the previous section while variation with respect to r gives terms of the form $\theta_3^8 \theta_{12}^5 / R^{13} r^8$. However, unlike the eight fermion term in $SU(2)$, such a variation is not in general invariant. It turns out that another two loop twelve fermion term can be constructed from (5.9) with the correct powers of R , r , θ_3 , and θ_{12} to vary under a supersymmetry transformation into the variation of (5.11). To derive such a term one

²In the discussion that follows, θ^8 is understood to represent the full tensor structure of the θ^8 term appearing in (5.2) after the derivatives have acted on $1/r^7$.

again starts with (5.9) and brings up one power of x_{12} from the denominator to combine with a θ_{12} from the numerator to make an $SU(2)$ coupling of the form,

$$[\theta_3^7 \theta^a (x^a \cdot \nabla_R)] \frac{1}{R^{11}}. \quad (5.12)$$

Making a second loop by contracting θ^a with x^a and attaching five θ_{12} 's gives about forty-five terms of the form $\theta_3^7 \theta_{12}^5 / R^{12} r^8$, some of which vary into $\theta_3^8 \theta_{12}^5 / R^{13} r^8$.

To constrain the θ_{12} tensor structure and functional dependence on r for the $\theta_3^8 \theta_{12}^4 / R^{13} r^7$ term, we would proceed by writing down the four terms compatible with $SO(9)$ invariance. However, we also need to consider the sixty or so possible tensor structures for the $\theta_3^7 \theta_{12}^5 / R^{12} r^8$ terms. In principle, one can imagine having enough constraints from the differential equations to fully determine the θ_{12} structure, but in practice the algebra becomes very cumbersome.

5.3 Twelve fermion term with the fewest factors of R ?

In the previous section, we encountered difficulties when there are two twelve fermion terms which transform into each other. It is natural to suppose that a twelve fermion term with the fewest powers of R is free of such difficulties. This term can be generated by starting with the four fermion term in (5.2) which we write schematically as $\theta^4 v^2 / r^9$. In $SU(3)$ there will be a term like $\theta_3^4 v^a v^a / R^9$. Contracting the $SU(2)$ fields v^a to form a light loop with eight fermions gives

$$\frac{\theta_3^4 \theta_{12}^8}{R^9 r^{11}}. \quad (5.13)$$

This term has the nice property that the θ_{12}^8 / r^{11} part can be shown to be exact following the argument presented in section 5.1. We also know that the θ_3^4 / R^9 structure is not

renormalized since it was derived from the exact effective operator $\theta_3^4 v^a v^a / R^9$ which is part of the supersymmetric completion of the non-renormalized operator θ_3^8 / R^{11} .

A potential problem arises with using (5.13) to prove a non-renormalization theorem in $SU(3)$ due to the fact that this term might not have the fewest powers of R if acceleration terms are present in the one-loop matrix model effective action. For example, a term of the form

$$\theta_3^3 \theta^a \dot{v}^a / R^8 \quad (5.14)$$

would lead to a two-loop term like $\theta_3^3 \theta_{12}^9 / R^8 r^{12}$. We would then encounter a proliferation of tensor structure in trying to constrain twelve fermion terms which vary into each other as in the previous section. There are a couple of reasons to suspect that such an acceleration term is not present in the matrix model effective action. In our calculation of the coefficient for v^4 in chapter two, we did not encounter acceleration terms. In addition, the R dependence of the corresponding velocity structure, $v_3^2 v_{12}^4 / R^6 r^8$ does not appear in the detailed calculation of three graviton scattering [30]. However, if such an acceleration term did exist, it is possible that it wouldn't upset the supersymmetric completion of v^4 by being invariant to supersymmetry transformations.

If it were true that acceleration terms of the form (5.14) were absent in the matrix model effective action we could generalize to $SU(4)$ to show that $\theta_4^4 \theta_3^8 / R_4^9 R_3^{11}$ is exact along with its supersymmetric completion $\theta_4^4 \theta_3^4 v^a v^a / R_4^9 R_3^9$. Contracting $\langle v^a v^a \rangle$ and attaching eight light θ_{12} 's to the loop gives

$$\theta_4^4 \theta_3^4 \theta_{12}^8 / R_4^9 R_3^9 r^{11}. \quad (5.15)$$

The θ_{12}^8 / r^{11} is again exact and we could continue this process to show the exactness of a certain class of θ^{4N} terms in $SU(N)$.

5.4 Outline of a Direct Proof for v^{2N} terms in $SU(N)$

In the previous two sections we have been unable to convincingly constrain twelve fermion terms without performing more detailed calculations. We will now outline a proof for non-renormalization theorems involving some v^{2N} terms in $SU(N)$ by investigating the implications for the propagator $\langle x^a x^a \rangle$ due to the successful proof in [48] for the non-renormalization of twelve fermion terms in $SU(3)$.

Before beginning we will need to assume that one could construct the complete supersymmetric completion for v^6 terms in $SU(3)$ and establish invariance to supersymmetry transformation in analogy with the one-loop supersymmetric completion of v^4 terms in $SU(2)$ [51], thereby firmly demonstrating that non-renormalization for θ^{12} terms implies the same for v^6 terms. Once this is done, we will know that the term of interest,

$$v_3^4 (\vec{v}_{12} \cdot \nabla_R)^2 \frac{1}{R^7} \frac{1}{r^5}, \quad (5.16)$$

is an exact result as expected from the matrix model agreement with supergravity for this term. Knowing that (5.16) is exact allows us to reach some conclusions about corrections to the propagator. In general the massive propagator for the $SU(2)$ modes has the form

$$\langle x^{+i} x^{-j} \rangle = \delta^{ij} (A + Bv^2 + C(\vec{r} \cdot \vec{v})^2) + Dv^i v^j + \dots \quad (5.17)$$

where ... indicates higher order velocity terms. Recall, in deriving (5.16), we began with the non-renormalized operators

$$\frac{v_{13}^4}{|\vec{x}_{13}|^7} + \frac{v_{23}^4}{|\vec{x}_{23}|^7} \quad (5.18)$$

to generate

$$v_3^4 (\vec{x}^a \cdot \nabla_R)^2 \frac{1}{R^7}, \quad (5.19)$$

and then used the $v^i v^j$ part of the propagator. Because (5.16) is exact we know that the coefficient D is also exact which gives us the potential to draw conclusions about the exactness of higher order terms that we generated in a similar fashion in chapter four. It is important to note, however, that we can only draw conclusions about the coefficient D in the propagator because none of the δ^{ij} pieces contribute to a two-loop term due to the fact that the Laplacian annihilates $1/R^7$. For this reason we will only be able to make a statement about $R \cdot v$ terms below.

To generalize to $SU(4)$, we would need to establish that the term

$$v_4^4(\vec{v}_3 \cdot \nabla_{R_4})^2 \frac{1}{R_4^7} \left(\frac{1}{|\vec{x}_{13}|^5} + \frac{1}{|\vec{x}_{23}|^5} \right) \quad (5.20)$$

is exact in analogy with the exactness of (5.16). This would seem to follow from the analysis carried out by [48] in constraining twelve fermion terms involving two elements of the Cartan sub-algebra and again assuming that the supersymmetric completion could be established. We could then proceed as before to generate

$$v_4^4(\vec{v}_3 \cdot \nabla_{R_4})^2 \frac{1}{R_4^7} (\vec{x}^a \cdot \nabla_{R_3})^2 \frac{1}{R_3^5}. \quad (5.21)$$

Now contracting the x^a 's and using the exact $v^i v^j$ part of the propagator we arrive at the non-renormalized term

$$v_4^4(\vec{v}_3 \cdot \nabla_{R_4})^2 \frac{1}{R_4^7} (\vec{v}_{12} \cdot \nabla_{R_3})^2 \frac{1}{R_3^5} \frac{1}{r_{12}^5}. \quad (5.22)$$

As eluded to above, a little bit of care needs to be exercised, since (5.22) contains terms involving

$$v_4^4(\vec{v}_3 \cdot \nabla_{R_4})^2 \frac{1}{R_4^7} (\vec{v}_{12} \cdot \vec{R}_3)^2 \frac{1}{R_3^9} \frac{1}{r_{12}^5} \quad (5.23)$$

as well as

$$v_4^4(\vec{v}_3 \cdot \nabla_{R_4})^2 \frac{1}{R_4^7} (\vec{v}_{12} \cdot \vec{v}_{12}) \frac{1}{R_3^7} \frac{1}{r_{12}^5}. \quad (5.24)$$

The term (5.24) can also be generated by taking the $\delta^{ij}Bv^2$ part of the propagator in going from (5.21) to (5.24), and the coefficient, B , could in principle be renormalized. As a consequence we could only make a statement about the terms appearing in (5.23).

In generalizing to $SU(N+1)$, we would have to make the plausible assumption that the v^{2N} velocity structure is exact in $SU(N+1)$, once it has been shown that the same velocity structure in $SU(N)$ is exact as we did for (5.20) above. One could then proceed to show that the velocity structure in $v^{2(N+1)}$ is not renormalized for any N .

Chapter 6

Conclusion

We have seen in the course of this thesis that the matrix model has successfully passed the cardinal test of reproducing multigraviton scattering in supergravity. Terms in three-graviton scattering which were not explicitly studied in the work of [30] are shown to agree. We have seen that terms in four-graviton and n -graviton scattering, for arbitrary n , also agree. The fact that all the matrix model terms which agree with supergravity are protected by non-renormalization theorems (or strong evidence for such theorems exists), further validates the conjecture that finite N matrix theory describes the DLCQ of M-theory with the DLCQ of supergravity as a low energy approximation.

In chapter 4, we encountered matrix model terms which did not appear to have a spacetime description in supergravity. For example, the v^6 term which gets renormalized in the $SU(4)$ matrix model effective action has the wrong N dependence to correspond to a tree level supergravity amplitude. It is possible that new issues arise at the level of four-graviton scattering. Previous work in two particle scattering showing a discrepancy between supergravity and the matrix model at finite N in a curved background [53, 53] can

be translated into a problem with four-graviton scattering. One can think of the two “far away” gravitons as providing a background for the scattering of the other two gravitons. It will be interesting to further investigate some of the matrix model terms describing four-graviton scattering that we calculated but did not compare with supergravity. If the four graviton scattering amplitude agrees for terms with differing tensor structures, it would seem natural to expect that all the leading v^{2N} velocity terms agree in $SU(N)$ for any N . The subleading terms, like the renormalized v^6 terms in $SU(4)$, presumably have an interpretation in M-theory in the large N limit.

In chapter 3 we discussed the problems with defining the matrix model effective action starting at two loops. Ordinarily one wants to make a Wilsonian type definition and integrate out massive states down to some scale. However, in the explicit two-loop calculation [25] both massive and massless states had to be integrated out to obtain the expected non-renormalization of the v^4 term. Integrating over the massless states in the matrix model leads to infrared divergences and an ill-defined effective action. We were able to show that for the v^4 terms that all the infrared divergences cancel due to the fact that the Laplacian in nine dimensions annihilates $\frac{1}{r^7}$. Consequently for terms that are generated from $\frac{1}{r^5}$, we expect infrared renormalizations. Terms with this behavior, such as

$$v_3^4 v_{12}^2 / R^9 r^5, \tag{6.1}$$

already exist at two loops and in chapter 4 we saw they agreed with the supergravity amplitude. We also know from the work in [48], the fermionic supersymmetric completion of such a term is not renormalized, so it seems natural to suspect that this velocity term is exact. Why then does (6.1) receive infrared renormalizations? The answer would seem to be that the coordinate space propagator should be defined so that it vanishes at equal times.

This is the correct prescription to reproduce the Born series in potential scattering and free the matrix model of infrared divergences. On the surface, this answer seems like a logical prescription, but the fact that an analogous term in $SU(4)$ receives finite renormalizations (as demonstrated at the end of chapter 4) makes one worry that perhaps we are missing another subtle lesson the matrix model is trying to teach us.

Throughout this thesis we have been mostly studying eleven dimensional DLCQ M-theory. In section 4.6.2 we very briefly considered up to four compact dimensions, indicating that our multigraviton scattering results should hold for DLCQ M-theory in as low as seven spacetime dimensions. Of course it would be nice to be able to make contact with four spacetime dimensions. However, remembering that M-theory compactified k dimensions is described by $k+1$ dimensional super Yang-Mills theory, one would expect trouble when $k=4$, due to non-renormalizable operators. It turns out that new states can be added to the field theory allowing an M-theory description in no less than six spacetime dimensions [19, 18]. Trying to define M-theory in 4 or even 5 spacetime dimensions requires new degrees of freedom that have yet to be understood [12].

Given the matrix model's tremendous success in reproducing supergravity amplitudes in addition to the numerous other non-trivial tests the matrix model has passed, it seems reasonable to suspect that the matrix model is a correct formulation for the non-perturbative limit of superstring theory.

Bibliography

- [1] J. Polchinski, Phys. Rev. Lett. **75**, 4724 (1995).
- [2] J. Polchinski, hep-th/9611050.
- [3] E. Witten, Nucl. Phys. **B 443**, 85 (1995).
- [4] P. K. Townsend, Phys. Lett. **B 350**, 184 (1995).
- [5] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory, Vols. 1 and 2* (Cambridge University Press, Cambridge, 1987).
- [6] J. Polchinski, *String Theory* (Cambridge University Press, Cambridge, 1998).
- [7] J. Wess and J. Bagger, *Supersymmetry and Supergravity* (Princeton University Press, Princeton, New Jersey, 1992).
- [8] J. Schwarz, hep-th/9607067.
- [9] E. Cremmer, B. Julia, and J. Scherk, Phys. Lett. **B 76**, 139 (1978).
- [10] T. Banks, W. Fischler, S. Shenker, and L. Susskind, Phys. Rev. **D 55**, 5112 (1997), hep-th/9610043.
- [11] E. Witten, Nucl. Phys. **B 460**, 335 (1996), hep-th/9510135.

- [12] T. Banks, Nucl. Phys. Proc. Suppl. **67**, 180 (1998), hep-th/9710231.
- [13] D. Bigatti and L. Susskind, hep-th/9712072.
- [14] W. Taylor, hep-th/9801182.
- [15] A. Bilal, Fortsch. Phys. **47**, 5 (1999), hep-th/9710136.
- [16] L. Susskind, hep-th/9704080.
- [17] H. C. Pauli and S. J. Brodsky, Phys. Rev. **D 32**, 1993 (1985).
- [18] N. Seiberg, Phys. Rev. Lett. **79**, 3577 (1997), hep-th/9710009.
- [19] A. Sen, Adv. Theor. Math. Phys. **2**, 51 (1998), hep-th/9709220.
- [20] M. R. Douglas, D. Kabat, P. Pouliot, and S. Shenker, Nucl. Phys. **B 485**, 85 (1997),
hep-th/9608024.
- [21] U. H. Danielsson, G. Ferretti, and B. Sundborg, Int. J. Mod. Phys. **A 11**, 5463 (1996),
hep-th/9603081.
- [22] D. Kabat and P. Pouliot, Phys. Rev. Lett. **77**, 1004 (1996), hep-th/9603127.
- [23] J. Plefka and A. Waldron, Nucl. Phys. **B 512**, 460 (1998), hep-th/9710104.
- [24] D. Berenstein and R. Corrado, Phys. Lett. **B 406**, 37 (1997), hep-th/9702108.
- [25] K. Becker and M. Becker, Nucl. Phys. **B 506**, 48 (1997), hep-th/9705091.
- [26] M. Peskin and D. Schroeder, *An Introduction to Quantum Field Theory* (Addison-Wesley, Reading Massachusetts, 1995).
- [27] C. Bachas, Phys. Lett. **B 374**, 37 (1996), hep-th/9511043.

- [28] B. S. DeWitt, Phys. Rev. **162**, 1239 (1967).
- [29] S. Sannan, Phys. Rev. **D 34**, 1749 (1986).
- [30] Y. Okawa and T. Yoneya, Nucl. Phys. **B 538**, 67 (1999), hep-th/9806108.
- [31] K. Becker, M. Becker, J. Polchinski, and A. Tseytlin, Phys. Rev. **D 56**, 3174 (1997), hep-th/9706072.
- [32] S. Paban, S. Sethi, and M. Stern, Nucl. Phys. **B 534**, 137 (1998), hep-th/9805018.
- [33] M. Dine and N. Seiberg, Phys. Lett. **B 409**, 239 (1997), hep-th/9705057.
- [34] Y. Okawa, hep-th/9903025.
- [35] M. Dine and A. Rajaraman, Phys. Lett. **B 425**, 77 (1998), hep-th/9710174.
- [36] R. Li, SCIPP preprint in preparation.
- [37] J. Plefka, M. Serone, and A. Waldron, Phys. Rev. Lett. **81**, 2866 (1998), hep-th/9806081.
- [38] K. Becker and M. Becker, Phys. Rev. **D 57**, 6464 (1998), hep-th/9712238.
- [39] I. W. Taylor and M. V. Raamsdonk, Phys. Lett. **B 438**, 248 (1998), hep-th/9806066.
- [40] J. McCarthy, L. Susskind, and A. Wilkins, Phys. Lett. **B 437**, 62 (1998), hep-th/9806136.
- [41] U. H. Danielsson, M. Kruczenski, and P. Stjernberg, Nucl. Phys. **B 540**, 295 (1999), hep-th/9807071.
- [42] M. Dine, R. Echols, and J. Gray, Phys. Lett. **B 444**, 103 (1998), hep-th/9805007.

- [43] M. Fabbrichesi, G. Ferretti, and R. Iengo, JHEP **9806**, 2 (1998), hep-th/9806018.
- [44] Y. Okawa and T. Yoneya, Nucl. Phys. **B 541**, 163 (1999), hep-th/9808188.
- [45] W. Taylor, Phys. Lett. **B 394**, 283 (1997), hep-th/9611042.
- [46] L. Susskind, hep-th/9611164.
- [47] S. Paban, S. Sethi, and M. Stern, JHEP **9806**, 12 (1998), hep-th/9806028.
- [48] S. Sethi and M. Stern, hep-th/9903049.
- [49] D. A. Lowe, JHEP **9811**, 9 (1998), hep-th/9810075.
- [50] J. A. Harvey, Nucl. Phys. Proc. Suppl. **68**, 113 (1998), hep-th/9706039.
- [51] S. Hyun, Y. Kiem, and H. Shin, hep-th/9903022.
- [52] M. Barrio, R. Helling, and G. Polhemus, JHEP **9805**, 12 (1998), hep-th/9801189.
- [53] M. R. Douglas, H. Ooguri, and S. H. Shenker, Phys. Lett. **B 402**, 36 (1997), hep-th/9702203.
- [54] M. Douglas, D. Kabat, P. Pouliot, and S. Shenker, Nucl. Phys. **B 485**, 85 (1997), hep-th/9608024.
- [55] M. R. Douglas and H. Ooguri, Phys. Lett. **B 425**, 71 (1998), hep-th/9710178.