Analysis Qualifying Exam

Spring 2008

This exam has five (5) questions. Please answer each part as completely as possible. Unsupported work will receive no credit, and partially completed work may receive partial credit. Each question is worth five (5) points, for a grand total of 25 points possible. Good luck to you all!

1. Suppose $K$ is a compact subset of $\mathbb{R}$ and $f : K \rightarrow (0, \infty)$ is a continuous function. Show there exists $\delta > 0$ such that for all $x \in K$, $f(x) \geq \delta$.

2. Let $f$ be a twice differentiable real-valued function defined on the interval $(a, b)$. Suppose that $a < x_1 < x_2 < x_3 < b$, $f(x_1) > f(x_2)$ and $f(x_3) > f(x_2)$. Prove there exists $c \in (a, b)$ such that $f''(c) > 0$.

3. (a) State a definition of what it means for a function $f$ to be Riemann integrable on a finite interval $[a, b]$.
   (b) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be the function defined by setting $f(x) = 0$ for irrational numbers $x$ and
   
   $$f\left(\frac{m}{n}\right) = \begin{cases} 
   7 & \text{if } n \text{ is even} \\
   5 & \text{if } n \text{ is odd}
   \end{cases}$$

   for relatively prime integers $m$ and $n$. Use the definition of Riemann integrable to prove that $f$ is not Riemann integrable on $[0, 1]$.

4. (a) Suppose $a_n$ is nonnegative for all $n \in \mathbb{N}$. Prove that if $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n^2$ converges.
   (b) Prove or give a counter-example to the following statement.

   If $a_n$ is a real number for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n^2$ converges.

5. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous real-valued functions defined on an interval $[a, b]$ and $f$ be a real-valued function defined on $[a, b]$.
   (a) Prove that if the sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly to $f$, then
   
   $$\lim_{n \to \infty} \int_{a}^{b} f_n(x) \, dx = \int_{a}^{b} f(x) \, dx$$

   (b) Does part (a) remain true if we replace “converges uniformly” with “converges pointwise”? Prove or give a counterexample.
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1. Let $E \subseteq \mathbb{R}$ be a nonempty bounded set.
   (a) Define $\text{sup}(E)$.
   (b) Assume $\text{sup}(E) \notin E$. Prove that there exists a strictly increasing sequence $(x_n)$ which converges to $\text{sup}(E)$ and $x_n \in E$ for each $n \in \mathbb{N}$.

2. Let $f : [0, \infty) \rightarrow \mathbb{R}$ be a function which is continuous on $[0, \infty)$ and differentiable on $(0, \infty)$. Suppose that $f(0) = 0$ and $f'$ is increasing. Let $g(x) = f(x)/x$ for $x > 0$. Prove that $g$ is increasing. Do not assume that $f$ is twice differentiable.

3. Classify all functions $f : [0, \infty) \rightarrow [0, \infty)$ which satisfy the following three conditions:
   (1) $f$ is Riemann integrable on any closed and bounded subinterval of $[0, \infty)$, (2) $f(x) > 0$ for all $x > 0$ and (3) $(f(x))^2 = \int_0^x f(t) \, dt$ for all $x > 0$.
   Hint: First prove that any such $f$ must be differentiable.

4. Let $(x_n)$ be a sequence of positive numbers. Prove that $\sum_{n=1}^{\infty} \frac{x_n}{1 + x_n}$ is convergent if and only if $\sum_{n=1}^{\infty} x_n$ is convergent. (You may use the well-known tests for convergence.)

5. Let $f(x) = \sum_{k=1}^{\infty} \frac{\cos(kx)}{k^3}$.
   (a) Prove that $f$ is Riemann integrable on $[0, \frac{\pi}{2}]$ and $\int_0^{\frac{\pi}{2}} f(x) \, dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k + 1)^4}$.
   (b) Prove that $f$ is differentiable.
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1. (a) Prove that an uncountable subset of \( \mathbb{R} \) must have an uncountable bounded subset.
(b) Use part (a) to deduce that every uncountable subset of \( \mathbb{R} \) has a limit point in \( \mathbb{R} \).

2. Let \( f \) be differentiable on \( \mathbb{R} \) with \( a = \sup \{|f'(x)| : x \in \mathbb{R}\} < 1 \). Select \( s_0 \in \mathbb{R} \) and define \( s_n = f(s_{n-1}) \) for \( n \geq 1 \). Thus \( s_1 = f(s_0), s_2 = f(s_1) \), etc. Prove that \( \{s_n\} \) is a convergent sequence.

3. (a) State a definition of Riemann integrable that makes use of partitions.
(b) Let \( a, b \in \mathbb{R} \) with \( a < b \). Let \( f : [a, b] \rightarrow \mathbb{R} \) and \( c \in (a, b) \). Suppose \( f \) is Riemann integrable on \([a, c]\) and on \([c, b]\). Using your definition from part (a), show \( f \) is Riemann integrable on \([a, b]\) and
\[
\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx.
\]

4. Suppose \( a_n \) and \( b_n \) are nonnegative for all \( n \in \mathbb{N} \).
(a) Prove that if \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} b_n \) both converge, then \( \sum_{n=1}^{\infty} a_n b_n \) also converges.
(b) Improve the result of part (a) by replacing the requirement of convergence of \( \sum_{n=1}^{\infty} b_n \) with something weaker.

5. (a) Give an explicit description of \( D = \{x \in \mathbb{R} : \sum_{n=1}^{\infty} \frac{n! x^n}{5^n} \text{ converges}\} \).
(b) Let \( f(x) = \sum_{n=1}^{\infty} \frac{n! x^n}{5^n} \) for \( x \in D \). For which \( x \in D \) is \( f \) continuous?
1. Let $A \subseteq \mathbb{R}$. Prove that the set of limit points of $A$ is closed.

2. Assume $(a_n)_{n=1}^\infty$ is a sequence of positive real numbers. Suppose $f$ is a differentiable function on $[0, \infty)$ with $f(0) = 0$ and $f'$ bounded. Prove that
   \[
   \text{if } \sum_{n=1}^\infty a_n \text{ converges, then } \sum_{n=1}^\infty f(a_n) \text{ converges absolutely.}
   \]

3. For each natural number $n$, let $f_n(x) = nx^n(1 - x)$.
   
   (a) Find the pointwise limit of $\{f_n\}$ on $[0, 1]$.
   
   (b) Does
   \[
   \lim_{n \to \infty} \int_0^1 f_n(x) \, dx = \int_0^1 \lim_{n \to \infty} f_n(x) \, dx?
   \]
   
   (c) Show that $\{f_n\}$ does not converge uniformly on $[0, 1]$.
   
   (Note: $\lim_{n \to \infty} \left( \frac{n}{n+1} \right)^{n+1} = e^{-1}$.)

4. (a) State a definition of the Riemann integral.
   
   (b) Assume $g$ and $f$ are nonnegative functions defined on $[a, b]$. Suppose $f$ is bounded and Riemann integrable on $[a, b]$, and $g$ is continuous on $[a, b]$. Prove directly from the definition of the Riemann integral that $fg$ is Riemann integrable on $[a, b]$.

5. (a) Define uniform continuity of a function $f$ on a set $A$.
   
   (b) Prove that $f(x) = \frac{1}{1+x^2}$ is uniformly continuous on $\mathbb{R}$. 


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1. Let $A$ be a nonempty subset of $\mathbb{R}$ that is bounded above. Suppose there exists $\delta > 0$ such that for all $x, y \in A$ with $x \neq y$ we have $|x - y| \geq \delta$. Prove that $A$ contains its supremum.

2. Let $\{x_n\}$ and $\{y_n\}$ be sequences of real numbers such that for all $n \in \mathbb{N}$,

   \[ y_n \neq 0 \quad \text{and} \quad \frac{x_n}{y_n} \rightarrow 1. \]

   Show that if either sequence is bounded, then $(x_n - y_n) \rightarrow 0$.

3. Let $\{f_n\}$ be a decreasing sequence of nonnegative, continuous functions on a closed, bounded set $S$. Show that if $f_n \rightarrow 0$ pointwise on $S$, then $f_n \rightarrow 0$ uniformly on $S$. You may not use Dini’s Theorem.

4. Suppose $f$ is bounded and Riemann integrable on $[a, b]$. Without appealing to measure theory, prove that $|f|$ is Riemann integrable on $[a, b]$.

5. Find the smallest natural number $R$ that makes the following assertion true:

   “For every infinitely differentiable function $f : \mathbb{R} \rightarrow \mathbb{R}$ having $R$ distinct real roots, the fifth derivative, $f^{(5)}$, has at least one real root.”

   Provide a proof, including an example to show your choice for $R$ is the smallest.
Analysis Qualifying Exam

Winter 2006

This exam has five (5) questions. Please answer each part as completely as possible. Unsupported work will receive no credit, and partially completed work may receive partial credit. Each question is worth five (5) points, for a grand total of 25 points possible. Good luck to you all!

1. For every \( n \in \mathbb{N} \), let \( A_n \subseteq \mathbb{R} \). Is the closure of the intersection of these sets equal to the intersection of their closures?

   i.e. Is \( \left[ \bigcap_{n=1}^{\infty} A_n \right] = \bigcap_{n=1}^{\infty} \overline{A_n} \)?

   If not, state the exact containment relationship between the two sets, and provide a counterexample to show equality need not hold.

2. Consider the function \( g \) defined by

   \[
   g(x) = \begin{cases} 
   \frac{\pi}{2} + x^2 \sin \left( \frac{1}{x} \right) & \text{if } x \neq 0 \\
   0 & \text{if } x = 0
   \end{cases}
   \]

   (a) Show that \( g \) is differentiable at \( x = 0 \).
   (b) Find \( g'(0) \).
   (c) Prove or disprove: If a function \( f : \mathbb{R} \to \mathbb{R} \) is differentiable and \( f'(c) \neq 0 \), then \( f \) is monotone on an interval containing \( c \).

3. Let \( \{a_j\} \subseteq \mathbb{R} \) such that \( \sum_{j=1}^{\infty} a_j = \frac{3\pi}{4} \). For every \( n \in \mathbb{N} \), define

   \[ T_n = \frac{1}{n} \sum_{j=1}^{n} S_j \text{ where for each } j \in \mathbb{N}, S_j = \sum_{k=1}^{j} a_k. \]

   Does \( \{T_n\}_{n=1}^{\infty} \) converge? If so, find its sum.

4. Suppose that \( f \) is bounded and Riemann integrable on an interval \([a, b]\) and that \( g \) is uniformly continuous on the range of \( f \). Without appealing to measure theory, prove that the composition \( g \circ f \) is Riemann integrable on \([a, b]\).

5. Let \( f : [0, 1] \to \mathbb{R} \) be non-negative and continuous. Show that

   \[
   \lim_{n \to \infty} \int_{0}^{1} f(x^n) \, dx = f(0).
   \]
1. Let \( \{A_n\}_{n=1}^{\infty} \) be a sequence of compact subsets of \( \mathbb{R} \). Assume that \( A_1 \cap A_2 \cap \ldots \cap A_n \neq \emptyset \) for each natural number \( n \). Show that \( \bigcap_{n=1}^{\infty} A_n \neq \emptyset \).

2. Let \( f \) be a function that is continuous on \([0,1]\) and differentiable on \((0,1)\). Show that if \( f(0) = 0 \) and \( |f'(x)| \leq |f(x)| \) for all \( x \in (0,1) \), then \( f(x) = 0 \) for all \( x \in [0,1] \).

3. Let \( f, g, \) and \( h \) be bounded real-valued functions on the closed interval \([0,1]\). Suppose that \( f \) and \( h \) are Riemann integrable on \([0,1]\), \( \int_{0}^{1} f(x) \, dx = \int_{0}^{1} h(x) \, dx \), and \( f(x) \leq g(x) \leq h(x) \) for all \( x \in [0,1] \). Show that \( g \) is Riemann integrable on \([0,1]\).

4. Let \( f : [a,b] \to \mathbb{R} \) be monotonically decreasing. Show that
\[
\lim_{x \to c^+} f(x)
\]
exists for each \( c \in (a,b) \).

5. Show that
\[
\int_{0}^{\pi} \left[ \sum_{n=1}^{\infty} \frac{n \cos \left( \frac{nx}{2} \right)}{n^{2n}} \right] \, dx = \frac{2\pi^2}{\pi^4 + 1}.
\]
Justify each step in your solution.
Analysis Qualifying Exam  

Winter 2005

This exam has five (5) questions. Please answer each part as completely as possible. Unsupported work will receive no credit, and partially completed work may receive partial credit. Each question is worth five (5) points, for a grand total of 25 points possible. Good luck to you all!

1. Let $K \subset U \subset \mathbb{R}$ with $U$ open and $K$ compact. Show that there exists $\varepsilon > 0$ such that $(k - \varepsilon, k + \varepsilon) \subset U$ for all $k \in K$.

2. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be Lipschitz of order $\alpha > 0$ if
   \[ |f(x) - f(y)| \leq |x - y|^\alpha \]
   for all $x, y \in \mathbb{R}$.

   (a) Show that if $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz of order 2, then $f$ is constant.

   (b) Find all $\alpha > 0$ such that a function that is Lipschitz of order $\alpha$ must be a constant function. (Part (a) shows that 2 is one such value of $\alpha$.)

   (c) Give a value of $\alpha > 0$ and an explicit function $f : \mathbb{R} \to \mathbb{R}$ such that $f$ is Lipschitz of order $\alpha$ yet not differentiable.

3. Let $f : [a, b] \to \mathbb{R}$ be bounded.

   (a) Define what it means for $f$ to be Riemann integrable in terms of upper and lower sums.

   (b) Use the definition in part (a) to show that if $f : [a, b] \to \mathbb{R}$ is continuous, then $f$ is Riemann integrable on $[a, b]$.

4. Let $\{a_n\}$ be a sequence of real numbers.

   (a) Define what it means for $\{a_n\}$ to converge to a limit $\alpha$.

   (b) Suppose $\{a_n\}$ converges to $\alpha$, $\{b_n\}$ converges to $\beta$, $b_n \neq 0 \ \forall n$, and $\beta \neq 0$. Show that $\{a_n/b_n\}$ converges to $\alpha/\beta$. 
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1. (a) Given a set \( S \subseteq \mathbb{R} \), define what it means for a point \( p \) to be an interior point of \( S \).

   (b) For \( S \subseteq \mathbb{R} \), let \( S^o = \{ p \in \mathbb{R} : p \) is an interior point of \( S \} \). Show that \( S^o \) is an open subset of \( S \).

2. Let \( \{a_n\} \) be a bounded sequence and set \( B = \{ p \in \mathbb{R} : p \leq a_n \) for infinitely many \( n \} \). Show that \( B \) is nonempty, bounded above and that there exists a subsequence \( \{a_{n_k}\} \) that converges to the least upper bound of \( B \).

3. Let \( f : [a, b] \rightarrow \mathbb{R} \) be bounded and let \( \mathcal{P} = \{ a = x_0 \leq x_1 \leq x_2 \leq \ldots \leq x_n = b \} \) be a partition of \( [a, b] \).

   (a) Define the upper and lower Riemann sums for \( f \) with this partition.

   (b) Given \( f : [a, b] \rightarrow \mathbb{R} \), characterize Riemann integrability of \( f \) in terms of upper and lower Riemann sums.

   (c) Show that if \( f \) is Riemann integrable on \( [a, b] \), then \(|f| \) is Riemann integrable on \([a, b]\) as well.

4. Suppose \( f \) is continuous on all of \([a, b]\) and differentiable at all points of \((a, b)\) except possibly at a single point \( x_0 \in (a, b) \). If \( \lim_{x \to x_0} f'(x) \) exists, show that \( f'(x_0) \) exists and

   \[ f'(x_0) = \lim_{x \to x_0} f'(x) \]

5. Let \( f_j : (0, \infty) \rightarrow \mathbb{R} \) be defined for each \( j = 1, 2, 3, \ldots \) by

   \[ f_j(x) = \frac{\sin jx}{j^3 \sqrt{x}} \]

   (a) Find the pointwise limit of \( \{f_j\} \) and determine whether or not the convergence is uniform.

   (b) Show that the series \( \sum_{j=1}^{\infty} f_j(x) \) converges pointwise to a function \( f \) on an interval containing \( \pi \).

   (c) Prove that \( f \) is differentiable at \( \pi \) and compute \( f'(\pi) \). (You may leave your answer in the form of a sum.)
1. Let $A$ be a closed subset of $\mathbb{R}$. Show there exists a function $f : \mathbb{R} \to \mathbb{R}$ such that: (i) $f$ is continuous and (ii) $f(x) = 0$ if and only if $x \in A$.

2. Let $K$ be a compact subset of $\mathbb{R}$, $B$ be a subset of $\mathbb{R}$ and $f : K \to B$ be a continuous, one-to-one function that maps onto the set $B$. Prove that $f^{-1} : B \to \mathbb{R}$ is continuous.

3. It is a fact that, given any continuous function $f$ on $[a, b]$ and any $\epsilon > 0$, there exists a polynomial $p$ such that $|f(x) - p(x)| < \epsilon$ for all $x \in [a, b]$. Use this fact to prove that if $f$ is continuous on $[a, b]$ and for every polynomial $p(x)$ we have that

$$\int_a^b f(x)p(x)\,dx = 0,$$

then $f(x) = 0$ for all $x \in [a, b]$.

4. (a) State the Weierstrass M-test.
(b) State and give a counterexample to the converse of the Weierstrass M-test. Prove your counterexample works.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Let $a$ and $b$ be real numbers such that $a < b$ and $f'(a) \neq f'(b)$. If $N$ is a number between $f'(a)$ and $f'(b)$, then there is a number $M \in (a, b)$ such that $f'(M) = N$. 
Problem 1 Let

$$\chi(x) = \begin{cases} 
1 & \text{if } x \in \mathbb{Q} \\
0 & \text{if } x \in \mathbb{R} - \mathbb{Q} 
\end{cases}$$

and define $f(x) = x^n \chi(x)$.

(a) Show that $f(x)$ is not continuous at any point $x \neq 0$.
(b) Show that $f(x)$ is differentiable at $x = 0$ and find $f'(0)$.

Problem 2 Prove that $\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{\pi}{n}\right)$ converges on the interval $[0, \pi]$.

Problem 3 Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and suppose that $\lim_{x \to \infty} f(x) = \lim_{x \to -\infty} f(x) = 1$. Let $K = \{x \mid f(x) = 0\}$. Show that $K$ is compact.

Problem 4 Suppose that $f(x)$ is differentiable on the bounded interval $(a, b)$. Suppose further that there is a constant $M$ such that $|f'(x)| < M$ for all $x \in (a, b)$. Prove that $\lim_{x \to a} f(x)$ exists and is finite.

Problem 5 Let $f_M(x) = \lim_{N \to \infty} (\cos(M \pi x))^{2N}$ be defined on an interval $[a, b]$.

(a) Find $g(x) = \lim_{M \to \infty} f_M(x)$.

(b) State a definition of the Riemann Integral and use it to prove or disprove: $g(x)$ is Riemann Integrable on $[a, b]$. 
Analysis Qualifying Exam Fall 2003

1. Use the definition of the derivative to show that

\[
f(x) = \begin{cases} 
  x^3 \sin \frac{1}{x} + 5 & \text{if } x \neq 0 \\
  5 & \text{if } x = 0 
\end{cases}
\]

is differentiable at \( x = 0 \).

2. Let \( \sum_{n=1}^{\infty} a_n \) be a series of real numbers. Show that if there exists a number \( 0 < \rho < 1 \) such that \( \sqrt{|a_n|} \leq \rho \) for all sufficiently large \( n \), then the series converges absolutely.

3. Use the definition of the Riemann Integral to show that if \( f: [a, b] \to \mathbb{R} \) is bounded on \([a, b]\) and continuous at all but finitely many points, then \( \int_{a}^{b} f(x)dx \) exists.

4. Suppose \( f: [a, b] \to [a, b] \) is differentiable on \((a, b)\) and that there exists a number \( \alpha \in (0, 1) \) such that \( |f'(t)| \leq \alpha \) for all \( t \in (a, b) \). Show that \( f \) has a unique fixed point in \([a, b]\); that is, show there is a unique point \( x \in [a, b] \) such that \( f(x) = x \).

Hint: Pick \( x_0 \in [a, b] \) and set \( x_{k+1} = f(x_k) \). Estimate \( |x_{k+1} - x_k| \).

5. (a) Suppose that \( X \) is a compact subset of \( \mathbb{R} \) and that \( f: X \to \mathbb{R} \) is continuous on \( X \). Prove that \( f \) is uniformly continuous on \( X \).

(b) Let \( A_N = \left[-\frac{1}{2N}, -\frac{1}{2N+1}\right] \) and \( B_N = \left[\frac{1}{2N+1}, \frac{1}{2N}\right] \) for all natural numbers \( N \) and define \( X = \{0\} \cup \bigcup_{N=1}^{\infty} (A_N \cup B_N) \).

Set

\[
f(x) = \begin{cases} 
  \frac{1}{N} & \text{if } x \in A_N, N \in \mathbb{N} \\
  -\frac{1}{N} & \text{if } x \in B_N, N \in \mathbb{N} \\
  0 & \text{if } x = 0 
\end{cases}
\]

Show that \( f \) is uniformly continuous on \( X \).
1. (a) State a definition of the Riemann integral.
   (b) Use your definition from (a) to prove that if \( f(x) \) and \( g(x) \) are both bounded, Riemann integrable, and defined on the finite interval \([a, b]\), then \( f(x)g(x) \) is also Riemann integrable on \([a, b]\).

2. Does \( \sum_{k=1}^{\infty} \sin^2(\pi (k + \frac{1}{k})) \) converge? Fully justify any inequalities that you use.

3. Suppose \( f(x) \) has the property that \( f \) is differentiable everywhere and \( |f'(x)| < 10 \) for all \( x \in \mathbb{R} \). Prove that \( f(x) \) is uniformly continuous on \( \mathbb{R} \).

4. Let \( f_n(x) = \frac{\tan^{-1}(\sqrt{n}x)}{\sqrt{n}} \) where \( n \geq 1 \)
   (a) Find \( f(x) = \lim_{n \to \infty} f_n(x) \) and show that \( f_n \) converges to \( f \) uniformly on \( \mathbb{R} \).
   (b) Show that the sequence \( f_n'(x) \) does not converge uniformly on \( \mathbb{R} \).
   Recall: \( \frac{d}{dx} \tan^{-1}(x) = \frac{1}{1+x^2} \)

5. Let \( X = \{7\} \cup \{7 + \frac{1}{n}\}_{n=1}^{\infty} \). Use the open cover definition of compactness to show that \( X \) is compact.
1. Suppose $a_i > 0$ for all $i \geq 1$. Show that $\sum_{i=1}^{\infty} a_i$ converges if and only if $\sum_{i=1}^{\infty} \frac{a_i}{1+a_i}$ converges.

2. For $n \geq 1$, set $f_n(x) = \frac{x^n}{n+1}$. Prove that the sequence $f_n(x)$ converges uniformly on $[0, 1]$ and the sequence $f'_n(x)$ does not converge uniformly on $[0, 1]$.

3. Let $f(x)$ be a continuous function on $(0, 1]$. Show that $f(x)$ is uniformly continuous on $(0, 1]$ if and only if $\lim_{x \to 0^+} f(x)$ exists.

4. Let $K$ be a compact subset of $\mathbb{R}$, $x$ a real number not in $K$. The distance from $x$ to $K$ is defined to be

$$d(x, K) = \inf \{|x - y| : y \in K\}.$$  

Prove that $d(x, K) > 0$.

5. Let $f(x)$ be a non-negative and bounded function on a finite interval $[a, b]$. Assume the set

$$\{x \in [a, b] : f(x) \geq \epsilon\}$$

is a finite set for all $\epsilon > 0$. Show that $f(x)$ is Riemann integrable on $[a, b]$. 


1. Prove that if $\sum_{n=1}^{\infty} a_n$ is absolutely convergent and $\{b_n\}$ is a convergent sequence with $\lim_{n\to\infty} b_n = b$, then $\sum_{n=1}^{\infty} a_n b_n$ is convergent.

2. Prove or disprove

\[
f(x) = \begin{cases} 
1 & \text{if } x \in \mathbb{Q} \\
\phi & \text{if } x \notin \mathbb{Q} 
\end{cases}
\]

is Riemann integrable on the interval $[0, 1]$. Here $\mathbb{Q}$ denotes the set of rational numbers.

3. Let $f_n(x) = \frac{x^n}{1+x^n}, 0 \leq x \leq 1$.

(a) Show that $f_n(x)$ converges uniformly to 0 on the interval $[0, a]$ for any $a, 0 < a < 1$.

(c) Does $f_n(x)$ converge uniformly on the interval $[0, 1]$? Justify your answer.

4. (a) Construct an infinite subset of $\mathbb{R}$ with no limit points. Here $\mathbb{R}$ denotes the set of real numbers.

(b) Construct an infinite subset of $\mathbb{R}$ with countably many limit points

(c) Construct a countable subset of $\mathbb{R}$ with uncountably many limit points

5. Suppose $f(x)$ is differentiable and $f'(x)$ is continuous on $\mathbb{R}$. Let $a < c < b$ and suppose $f(a) < f(b) < f(c)$. Show that $f'(x) = 0$ for some $x \in [a, b]$. 

\[\text{Not Enough Information}\]
1. Let \( f(x) \) be a differentiable function on \((0, \infty)\) and suppose
\[
L = \lim_{x \to \infty} f'(x)
\]
eexists and is finite. Prove that if \( \lim_{x \to \infty} f(x) \) exists and is finite, then \( L = 0 \).

2. Suppose that \( \sum_{k=1}^{\infty} a_k \) converges absolutely. Prove that \( \sum_{k=1}^{\infty} |a_k|^p \) converges for all \( p \geq 1 \).

3. Let
\[
f(x) = \sum_{n=1}^{\infty} \frac{1}{1 + n^2 x^2}.
\]
(a) Determine the values of \( x \) for which the series converges. You must justify your results.
(b) Determine whether the convergence is uniform or not on the set \( S \):
\[
S = \left\{ x : \sum_{n=1}^{\infty} \frac{1}{1 + n^2 x^2} \text{ converges} \right\}.
\]
You must justify your answer.

4. Assume \( f(x) \) is a bounded Riemann integrable function on \([a, b]\). Set
\[
F(x) = \int_{a}^{x} f(x) \, dx.
\]
Use \( \epsilon - \delta \) notations to prove that \( F(x) \) is uniformly continuous on \([a, b]\).

5. Use the open covering definition of a compact set to show that
\[
E = \{ 0 \} \cup \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \}
\]
is a compact subset of \( \mathbb{R} \).
1. Show that if $a_n > 0$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} a_n$ is convergent, then $\sum_{n=1}^{\infty} \ln(1 + a_n)$ is convergent.

2. (a) Show that a finite intersection of open subsets of the real line $\mathbb{R}$ is open.
   (b) Give an example to show that an infinite intersection of open sets is not necessarily open.

3. Let $f(x)$ be a continuously differentiable function on the interval $[0, 10]$. Assume $|f'(x)| \leq M$ for $x \in [0, 10]$ and $f(0) = 0$. Prove that
   \[ \int_0^{10} f(x) \, dx \leq 50M. \]

4. (a) Show that $\sum_{n=1}^{\infty} \frac{x^n}{n^2 + e^x}$ converges for every $x \in \mathbb{R}$.
   (b) Show that the convergence is not uniform on $\mathbb{R}$.

5. Let $f : [a, b] \rightarrow [c, d]$ be onto and strictly increasing. Show that the inverse, $f^{-1}$, is continuous.
   (Hint: use a picture to help construct a proof.)
Do all five problems.

1. Let $x_k > 0$. Show that $\sum_{k=1}^{\infty} x_k$ and $\sum_{k=1}^{\infty} \frac{x_k}{1+x_k}$ either both converge or both diverge.

2. Use the open covering definition of compactness to show that any finite union of compact sets is compact.

3. Let $g$ be continuous on $[a, b]$. Let $\{f_n\}$ be a sequence of continuous functions on $[a, b]$ and let $f_n \to f$ uniformly on $[a, b]$. Show that
\[
\lim_{n \to \infty} \int_a^b f_n(x)g(x)\,dx = \int_a^b f(x)g(x)\,dx.
\]

4. Prove that if $f$ is differentiable on $\mathbb{R}$ and $\lim_{x \to \infty} f'(x) = 0$ then $f$ is uniformly continuous on $[0, \infty)$.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be onto and strictly increasing. Prove that $f$ is continuous.
ANALYSIS QUALIFYING EXAM
May 2000

Do all five problems.

1. Let \( K = \{1\} \cup (1 - \frac{1}{n})_{n=1}^{\infty} \). Use the open cover characterization of compactness to prove that \( K \) is compact.

2. Use the Mean Value Theorem to show that

\[ \sqrt{1 + z} < 1 + \frac{z}{2} \]

for all \( z > 0 \).

3. Let \( \mathbb{R} \) denote the real numbers. Let \( f : \mathbb{R} \to \mathbb{R} \) be differentiable at \( z = 0 \). Let \( f(0) = 1 \) and let \( f(x + y) = f(x)f(y) \) for all \( a, b \in \mathbb{R} \).

   (a) Show that \( f \) is differentiable on \( \mathbb{R} \).

   \[ \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{f(x) + f(h) - 1}{h} = \lim_{h \to 0} \frac{f(x) - 1}{h} \]

   \[ = f'(x) \cdot f(0) = f(x) \cdot f'(0) \]

   (b) Evaluate \( f'(x) \).

4. Let

\[ f(x) = \begin{cases} 0, & \text{when } x = 0 \\ \frac{x}{2^n}, & \text{when } \frac{1}{2^n} < x \leq \frac{1}{2^{n-1}}, \ n = 1, 2, 3, \ldots \end{cases} \]

Is \( f \) integrable on \([0, 1]\)? Prove your answer is correct.

5. Let \( \mathbb{R} \) denote the real numbers. For each natural number \( n \) and each number \( x \in (0, 1) \), define

\[ f_n(x) = \frac{1}{nx + 1} \]

Find the function \( f : (0, 1) \to \mathbb{R} \) to which the sequence \( \{f_n : (0, 1) \to \mathbb{R}\} \) converges pointwise. Is the convergence uniform? Prove your answer is correct.
Do all five problems.

1. Let \( \{a_n\}_{n=1}^{\infty} \) be a sequence of non-negative real numbers. Assume there exists a real number \( k > 1 \) such that the sequence \( \{n^k a_n\}_{n=1}^{\infty} \) converges. Show that the series \( \sum_{n=1}^{\infty} a_n \) converges.

2. Suppose that \( f \) is a continuous real-valued function on \([0, 1]\) such that \( f(x) \geq 0 \) for all \( x \) in \([0, 1]\) and \( f(1/2) > 0 \). Show that \( \int_0^1 f(x)dx > 0 \).

3. Show that the sequence of functions defined by

\[
f_n(x) = \frac{n}{x} e^{-\frac{x}{n}}.
\]

converges uniformly on any finite interval \((0, a]\) but not on the infinite interval \((0, \infty)\).

4. Assume \( a, b, \) and \( c \) are real numbers with \( a < b < c \). Let \( f \) be a real-valued function which is uniformly continuous on \((a, b)\) and on \([b, c]\). Show that \( f \) is uniformly continuous on \((a, c)\).

5. Let \( \{C_n\}_{n=1}^{\infty} \) be a sequence of compact subsets of the real numbers. Let

\[
A_n = C_1 \cap \ldots \cap C_n.
\]

Assume each \( A_n \) is nonempty. Show that \( \cap_{n=1}^{\infty} C_n \) is nonempty.
M.S. Analysis Examination

Please start each problem on a separate sheet of paper. Do all five.

1. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of real numbers. Prove that $\{a_n\}_{n=0}^{\infty}$ converges if and only if $\sum_{n=0}^{\infty} (a_{n+1} - a_n)$ converges.

2. Suppose that the sequence $\{f_n\}_{n=1}^{\infty}$ of real-valued functions converges uniformly on an interval $I$ to a bounded function $f$. Show that there is an $M > 0$ and a positive integer $N$ such that $|f_n(x)| < M$ for all $n > N$ and all $x \in I$.

3. A real number $c$ is a cluster point of a set $A \subseteq \mathbb{R}$ iff each neighborhood of $c$ contains an infinite subset of $A$. For $A \subseteq \mathbb{R}$, let $A'$ denote the set of cluster points of $A$. Show that there is no set $A \subseteq \mathbb{R}$ such that $A' = \mathbb{Q}$ where $\mathbb{Q}$ is the set of all rational numbers.

4. Using properties of the Riemann integral, show that if $f$ is Riemann integrable on $[a,b]$ and $x_0 \in [a,b]$, then

$$
\varphi(x) = \int_{x_0}^{x} f(t) \, dt
$$

is uniformly continuous on $[a,b]$.

5. Let $f$ be a real-valued differentiable function on the open interval $(0,2)$ whose derivative $f'$ is bounded on $(0,2)$. Show that the sequence $\{f(1/n)\}_{n=1}^{\infty}$ converges.
Please start each problem on a separate sheet of paper. Do all five.

#1. Suppose \( x \in E \subseteq \mathbb{R} \).
   a) Define: \( x \) is a limit point of \( E \).
   b) Define: \( x \) is an isolated point of \( E \).
   c) Suppose \( E \subseteq \mathbb{R} \) is uncountable, and \( L = \{ x \in E : x \) is a limit point of \( E \} \).
      Prove that \( L \) is uncountable.

#2. Let \( \{ f_n \}_{n=1}^{\infty} \) be a sequence of real-valued differentiable functions on \( \mathbb{R} \) that converges pointwise on \( \mathbb{R} \) to a function \( f \). Assume that there is a constant \( M \) such that for each positive integer \( n \), \( \left| f'_n(x) \right| \leq M \) for all \( x \in \mathbb{R} \). Prove that \( f \) is uniformly continuous on \( \mathbb{R} \).

#3. Let \( f, g \) and \( h \) be real valued functions on the closed interval \([a, b]\) such that \( f \) and \( g \) are integrable on \([a, b]\), \( \int_{a}^{b} f(x) \, dx = \int_{a}^{b} g(x) \, dx \) and \( f(x) \leq h(x) \leq g(x) \) for all \( x \in [a, b] \).
    Show that \( h \) is integrable on \([a, b]\).

#4. Let \( \{ a_n \} \) be an infinite sequence of positive real numbers, and define
    \[ P_n = \prod_{i=1}^{n} a_i = a_1 a_2 \ldots a_n \] for each \( n \in \mathbb{N} \). Consider the following conjecture:
    \[ \sum_{n=1}^{\infty} a_n \] converges iff \( \sum_{n=1}^{\infty} P_n \) converges. Prove or give a counterexample to each direction of the conjecture.

#5. For each \( n \in \mathbb{N} \), define \( f_n(x) = \cos \left( \frac{x}{n} \right)^2 \). Does \( f_n(x) \) converge for each \( x \in [0, \pi] \)?

If so, what is the limit function, and is the convergence uniform?
M.S. ANALYSIS QUALIFYING EXAM
September 16, 1998

Turn in four of the five problems below.

1. Assume that \( f \) is a continuous function on \([a, b]\) such that \( f(a) = f(b) = 0 \) and \( f(x) \geq 0 \) for all \( x \in [a, b] \). Prove that if \( \int_a^b f(x) \, dx = 0 \), then \( f(x) = 0 \) for all \( x \in [a, b] \).

2. Let \( \mathbb{R} \) denote the set of real numbers and let \( f : A \rightarrow \mathbb{R} \) be continuous where \( A \subset \mathbb{R} \) is compact. Show that \( f \) is uniformly continuous on \( A \).

3. Either prove or give a counterexample to the following assertion. If \( f \) and \( f_n, n = 1, 2, 3, \ldots \), are real valued continuous functions defined on \([0, 1]\) such that for each \( x \in [0, 1] \), \( \lim_{n \to \infty} f_n(x) = f(x) \), then \( f_n \) converges uniformly to \( f \) on \([0, 1]\).

4. Let \( f \) be a real valued function defined on \((a, b)\) whose derivative \( f' \) exists at each point of \((a, b)\). Show that for any \( c \in (a, b) \) there is a sequence \( x_n \) in \((a, b)\) such that for all \( n, x_n \neq c \) and \( \lim_{n \to \infty} f'(x_n) = f'(c) \).

5. Let \( a_n \) be a convergent sequence of real numbers such that \( \lim_{n \to \infty} a_n = a \) where \( a \neq 0 \). Show that there exist a real number \( h > 0 \) and an integer \( N > 0 \) such that for all \( i > N, h < |a_i| \).
Please start each problem on a separate sheet of paper. Do all five.

1. a. Define what is meant by a **connected set** in $\mathbb{R}$.

   b. Suppose that $f: \mathbb{R} \to \mathbb{R}$ is continuous.
      Prove that if $A \subseteq \mathbb{R}$ is connected, then $f(A)$ is connected.

   c. Prove that the converse of (b) is false.

2. Suppose that $f: \mathbb{R} \to \mathbb{R}$ is differentiable, $f(0) = 3$, $f(2) = 5$, and for every $x \in [0,2]$, $|f'(x)| \leq 1$.
   Prove that for every $x \in [0,2]$, $f(x) = x + 3$.

3. Find the function $f$ and the constant $k$ such that for every $x$,
   $\int_0^x f(t) \, dt = \int_0^x t^2 f(t) \, dt + 8x^6 + 6x^3 + k$.

4. Suppose that $\sum_{n=1}^{\infty} a_n$ is a convergent series of non-negative terms.
   For each of the following series, either prove that the series converges, prove that the series diverges, or prove that convergence or divergence cannot be determined from the given information.
   a. $\sum_{n=1}^{\infty} \sin(a_n)$  
   b. $\sum_{n=1}^{\infty} \cos(a_n)$

5. Let $\{r_i\}$ be an enumeration of all rational numbers in the interval $(0,1]$.
   For each positive integer $i$, define $f: [0,1] \to \mathbb{R}$ by $f_i(x) = \begin{cases} \frac{1}{i^2} & \text{if } 0 \leq x \leq r_i \\ 0 & \text{if } r_i < x \leq 1 \end{cases}$
   and define $f: [0,1] \to \mathbb{R}$ by $f(x) = \sum_{i=1}^{\infty} f_i(x)$.
   Is $f$ Riemann-integrable on the interval $[0,1]$?
   Prove your answer.
Turn in only four of the following five problems.
Each problem should begin on a separate sheet of paper.

1. Let \( \{A_\alpha\} \) be an uncountable collection of open subsets of \( \mathbb{R} \) and let \( A = \bigcup_{\alpha} A_\alpha \).

Prove that there is a countable collection \( \{I_n\} \) of open intervals satisfying both of the following:
   
   (i) \( \bigcup_{n=1}^{\infty} I_n = A \), and
   
   (ii) For every positive integer \( n \), there is an \( \alpha \) such that \( I_n \subseteq A_\alpha \).

2. Let \( a < x_0 < b \) and suppose the function \( f:(a,b) \to \mathbb{R} \) is differentiable on \( (a,b) \). If \( \lim_{x \to x_0} f'(x) = L \), prove that \( f'(x_0) = L \).

3. Suppose that \( f: \mathbb{R} \to \mathbb{R} \) is continuous everywhere.

Prove that \( \int_{0}^{x} f(u)(x-u)du = \int_{0}^{x} \left( \int_{0}^{u} f(t)dt \right) du \).

4. For any sequence \( \{a_n\} \) of real numbers, define
   
   \[ a_n^+ = \frac{a_n + |a_n|}{2} \quad \text{and} \quad a_n^- = \frac{a_n - |a_n|}{2} . \]

If the series \( \sum_{n=1}^{\infty} a_n \) converges conditionally, prove that \( \sum_{n=1}^{\infty} a_n^+ \) and \( \sum_{n=1}^{\infty} a_n^- \) both diverge.

5. Determine whether or not the sequence of functions \( g_n(x) = \frac{1}{x} e^{-\frac{n}{x}} \) converges uniformly on \( (0,\infty) \).
1. If \( A \subseteq R \), then either of the following two conditions can be used to define what is meant by \( A \) being closed.

   a. The complement of \( A \) is an open set.

   b. \( A \) contains all of its limit points.

   Prove that condition (a) implies condition (b).

2. Let \( \{r_k\} \) be an enumeration of the rationals. Define \( f: R \to R \) by

   \[
   f(x) = \begin{cases} 
   \frac{1}{k} & \text{if } x = r_k \\
   0 & \text{otherwise}
   \end{cases}
   \]

   Prove that \( f \) is continuous at \( x \) if and only if \( x \) is irrational.

3. Suppose that \( \{f_n\} \) is a sequence of real-valued Riemann -integrable functions on the interval \( [0,1] \) such that \( \lim_{n \to \infty} f_n \) exists and is finite everywhere.

   Prove or disprove:

   \[
   \lim_{n \to \infty} \int_0^1 f_n(x) \, dx = \int_0^1 \left( \lim_{n \to \infty} f_n(x) \right) \, dx
   \]

4. For each positive integer \( n \), define \( a_n = \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n} \).

   Determine whether or not the sequence \( \{a_n\} \) converges.

5. Let \( \{f_n\} \) be a sequence of real-valued functions on \( R \) that converges uniformly to a bounded, real-valued function \( f \) on \( R \).

   Prove that there exists a positive constant \( M \) and a positive integer \( N \) such that if \( n > N \), then \( |f_n(x)| \leq M \) for all \( x \in R \).
1. Prove that the sequence \( \{a_n\} \) defined by \( a_n = \left( \frac{n}{n+1} \right)^n \) converges, and find its limit.

2. Let \( f: \mathbb{R} \to \mathbb{R} \) and let \( x_0 \in \mathbb{R} \).

   Definition #1: \( f \) is called continuous at \( x_0 \) if for every \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that \( |f(x) - f(x_0)| < \varepsilon \) whenever \( 0 < |x - x_0| < \delta \).

   Definition #2: \( f \) is called continuous at \( x_0 \) if for every sequence \( \{x_n\} \) which converges to \( x_0 \), the sequence \( \{f(x_n)\} \) converges to \( f(x_0) \).

   Prove that if \( f \) is continuous at \( x_0 \) by definition #2, then it is also continuous at \( x_0 \) by definition #1.

3. Suppose that \( f \) is a non-negative continuous function defined on the interval \([0,1]\), and that there is a decreasing sequence \( \{a_n\} \) with \( \lim_{n \to \infty} a_n = 0 \) such that for all \( n \in \mathbb{Z}^+ \), \( \int_0^{a_n} f(x) \, dx = \int_{a_n}^1 f(x) \, dx \).

   Prove that \( f(x) = 0 \) for all \( x \in [0,1] \).

SEE NEXT PAGE FOR #4 AND #5.
M.S. Analysis Examination

March 1997

1. Recall that a subset $A$ of $\mathbb{R}$ is defined to be compact iff every open cover of $A$ contains a finite subcover of $A$. Prove directly from this definition of compact that every closed subset of a compact set is also compact.

2. Recall that a function $f: \mathbb{R} \to \mathbb{R}$ is called even iff $f(-x) = f(x)$ for all $x$. Suppose that $f: \mathbb{R} \to \mathbb{R}$ is an even function which is differentiable at $x = 0$. Prove that $f'(0) = 0$.

3. Give an example of a function $f$ such that $f$ is not Riemann-integrable on the interval $[0, 1]$, but $|f|$ is. Prove that your example is correct.

4. Define the sequence $\{a_n\}$ as follows. $a_1 = 2$ and $a_{n+1} = 2 - \frac{1}{a_n}$.

Prove that the sequence $\{a_n\}$ converges, and find its limit.

5. a. Prove that the sequence $f_n(x) = \cos\left(\frac{x}{n}\right)^2$ converges uniformly to $f(x) = 1$ on the interval $[0, \pi]$.

b. Compute $\lim_{n \to \infty} n \int_{0}^{\pi} \cos(x)^2 \, dx$. Justify your answer.
1. Let $f, g$ be continuous functions on $[0, 1]$. Suppose that $f(x) = g(x)$ for all rational $x \in [0, 1]$. Show that $f = g$.

2. Let $f$ be continuous on $[0, 1]$. Suppose $\int_0^x f(t) \, dt = \int_1^x f(t) \, dt$ for all $x$, $0 < x < 1$. Determine $f$.

3. Let $\{x_n\}$ be a sequence of positive numbers. Show that the series
\[
\sum_{n=1}^{\infty} x_n \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{x_n}{1 + x_n}
\]
both converge or both diverge.

4. Consider the sequence of functions $f_n(x) = xe^{-nx}$, $x \geq 0$.
   a) Prove that the sequence $f_n$ converges.
   b) Determine whether or not the convergence is uniform and prove your assertion.

5. Let $f : [0, \infty) \to \mathbb{R}$ be differentiable on $(0, \infty)$. If $f'(x) \to b \in \mathbb{R}$ as $x \to +\infty$, show that for any $h > 0$,
\[
\lim_{x \to +\infty} \frac{f(x + h) - f(x)}{h} = b.
\]
1. Define \( f : \mathbb{R} \to \mathbb{R} \) by \( f(x) = \begin{cases} x^2 & \text{if } x \text{ rational} \\ 0 & \text{if } x \text{ irrational} \end{cases} \). Prove that \( f \) is differentiable at \( x = 0 \).

2. If \( S_n = \int_0^{\pi/2} (\sin x)^n dx \), prove that the sequence \( \{S_n\} \) converges. Hint: Prove that \( \{S_n\} \) is monotone decreasing.

3. Prove the following Nested Interval Theorem. If \( \{I_n\} \) be a collection of closed, bounded and non-empty intervals in \( \mathbb{R} \) such that \( I_1 \supset I_2 \supset I_3 \supset \ldots \), then \( \bigcap_{n=1}^{\infty} I_n \neq \emptyset \).

4. Prove that \( \sum_{k=1}^{\infty} \frac{\cos \sqrt{k} x}{k^2} \) converges uniformly on \( \mathbb{R} \) to a function that is differentiable on \( \mathbb{R} \).

5. Let \( f : (a, b) \to \mathbb{R} \) be uniformly continuous. Prove that \( f \) can be extended to a continuous function on the closed interval \([a, b] \).
1. Let $a_n \geq 0$ and let $\lim_{n \to \infty} n^k a_n = 0$ for some $k > 1$.

Show that $\sum_{n=1}^{\infty} a_n$ converges.

2. Let $f: [a, b] \to \mathbb{R}$ be a monotone increasing function.
Show that $f$ is Riemann integrable on $[a, b]$.

3. Let $A \subseteq \mathbb{R}$ and let $(f_n)$ be a sequence of real-valued functions on $A$ converging uniformly
to a bounded real-valued function on $A$. Show that there are constants $K$ and $N$ such that
$|f_n(x)| \leq K$ for all $x \in A$ whenever $n > N$.

4. Let $f: \mathbb{R} \to \mathbb{R}$ be a function satisfying the condition:
$f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.

   a) Find $f(0)$.
   
   b) Show that if $\lim_{x \to 0} \frac{f(x)}{x}$ exists and is finite, then $f$ is differentiable on $\mathbb{R}$.

5. Let $E$ be a non-empty bounded subset of $\mathbb{R}$ and let $b = \sup E$ ($= \text{lub } E$).
Show that there exists a sequence $(x_n) \subseteq E$ such that $\lim_{n \to \infty} x_n = b$. 
1. a) State the definition of convergence of a series \( \sum_{n=1}^{\infty} a_n \).

b) Provide examples for each of the following (you need not prove that the examples satisfy the conditions):
   
   i) A series that converges.
   ii) A series that diverges.
   iii) A series that converges conditionally.
   iv) Divergent series \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} b_n \) for which \( \sum_{n=1}^{\infty} (a_n + b_n) \) converges.

2. Show that \( f(x) = \sin \sqrt{x} \) is uniformly continuous on \( [0, \infty) \).
   You may assume \( \sin x \) is differentiable on \( (-\infty, \infty) \).

3. Let \( f, g : \mathbb{R} \rightarrow \mathbb{R} \). Let \( f \) be differentiable at \( x_0 \), \( g \) be continuous at \( x_0 \), and \( f(x_0) = 0 \).
   Show that the product \( fg \) is differentiable at \( x_0 \).

4. Let \( K = \{0\} \cup \{\frac{1}{n} : n = 1, 2, 3, \ldots\} \) be a subset of \( \mathbb{R} \). Using the open cover definition of compactness, show that \( K \) is compact.

5. Show that \( \lim_{n \to \infty} \int_{0}^{\pi} \frac{\sin nx}{nx} \, dx \) exists and evaluate this limit.
   You may use \( \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \) without proof.
1. Using only the definition of a Cauchy sequence, show that every Cauchy sequence \( \{a_n\} \) in \( \mathbb{R} \) is bounded.

2. Let \( f_k : D \to \mathbb{R}, \ D \subset \mathbb{R}, \) and let \( |f_k(x)| \leq M_k \) for all natural numbers \( k \) and \( x \in D \).

   If \( \sum_{k=1}^{\infty} M_k \) converges, show that \( \sum_{k=1}^{\infty} f_k \) converges uniformly on \( D \).

3. Let \( f : [0,1] \to \mathbb{R} \) be continuous, \( f(0) = 0 \), and \( f'(x) \) exist for all \( x \in (0,1) \). Let \( f'(x) \) be increasing on \( (0,1) \).

   Show that \( g(x) = \frac{f(x)}{x} \) is increasing on \( (0,1) \).

4. Let \( f : \mathbb{R} \to \mathbb{R} \) be uniformly continuous and let \( g_k(x) = f(x + \frac{1}{k}) \) for all natural numbers \( k \).

   Show that \( \{g_k\} \) converges uniformly on \( \mathbb{R} \).

5. Let \( f : [0,1] \to \mathbb{R} \) be continuous. Show that \( \lim_{n \to \infty} \int_{0}^{1} f(x^n) \, dx = f(0) \).
Analysis Exam

Spring 1994

Do any 4 of the following 5 problems.
Turn in only those 4 that you want graded.

1. Let $a_n \geq 0$ and assume that there exists a $k > 1$ such that \( \lim_{n \to \infty} n^k a_n \) exists and is finite.

   Show that \( \sum_{n=1}^{\infty} a_n \) converges.

2. Define \( f(x) \) for \( 0 \leq x \leq 1 \) as follows:

   \[
   f(0) = 0
   \]

   For \( \frac{1}{2} < x \leq 1 \), \( f(x) = 1 \).

   For \( \frac{1}{3} < x \leq \frac{1}{2} \), \( f(x) = \frac{1}{2} \).

   In general, for \( \frac{1}{n+1} < x \leq \frac{1}{n} \), \( f(x) = \frac{1}{n} \) for \( n = 1, 2, 3, \ldots \).

   Show that \( f \) is Riemann integrable on \([0,1]\).

3. Show that the sequence \( f_n(x) = \frac{x}{x+n} \) is not uniformly convergent on the interval \([0,\infty)\).

4. a) Let \( f : E \to \mathbb{R} \) be a uniformly continuous function and let \( \{a_k\} \) be a Cauchy sequence in \( E \) where \( E \subset \mathbb{R} \). Prove that \( \{f(a_k)\} \) is also a Cauchy sequence.

   b) Give an example to show that the conclusion in part (a) is false if we only assume that \( f \) is continuous. You must justify your answer.

5. Let \( f \) be continuous on \([0,1]\) and differentiable on \((0,1)\). Let \( f(0) = 0 \) and \(|f'(x)| \leq |f(x)|\) for all \( x \in (0,1) \). Prove that \( f(x) = 0 \) on \([0,1]\).
Analysis Exam
Winter 1994

Do any 4 of the following 5 problems.
Turn in only those 4 which you want graded.

1. Suppose \( \sum_{j=1}^{\infty} a_j \) converges absolutely. Let \( s_k = \sum_{j=1}^{k} a_j \). Prove that \( \sum_{j=1}^{\infty} a_j s_j \) converges absolutely.

2. Let \( S \) be an uncountable subset of \( \mathbb{R} \). Prove that \( S \) has an uncountable number of accumulation points.

3. Suppose \( P : [0,2] \rightarrow \mathbb{R} \) is differentiable with \( P(0) = 1 \), \( P(2) = 3 \), and \( |P'(x)| \leq 1 \) for all \( x \in [0,2] \). Show \( P(x) = x + 1 \).

4. Let \( f \) and \( g \) be continuous real-valued functions on an interval \([a,b]\). Suppose

\[
\int_{a}^{b} f(t) \, dt = \int_{a}^{b} g(t) \, dt.
\]

Show there exists \( c \in (a,b) \) such that \( f(c) = g(c) \).

5. Suppose that for each \( k = 1, 2, 3, \ldots \), \( f_k : \mathbb{R} \rightarrow \mathbb{R} \) is continuous and \( f_k \) converges uniformly to \( f \) on \( \mathbb{R} \). If \( \lim_{k \to \infty} x_k = x \), show \( \lim_{k \to \infty} f_k(x_k) = f(x) \).
Analysis Exam

Spring 1993

Do any 4 of the 5 problems. Hand in only the 4 problems you wish to have graded.

1. Using the open cover definition of compactness, prove that a compact subset of $\mathbb{R}$ is closed and bounded.

2. Assume $\lim_{x \to \infty} f'(x) = 0$, and find

$$\lim_{x \to \infty} (f(x + 1) - f(x)).$$

3. Let $f : [a, b] \to \mathbb{R}$ be a monotone increasing function. Prove that $f$ is Riemann integrable.

4. Prove that $\sum a_n$ converges absolutely if and only if $\sum a_n b_n$ converges for every bounded sequence $(b_n)$.

5. Let $f_n(x) = n x^n (1 - x)$, $0 \leq x \leq 1$.
Show that $\{f_n\}$ converges pointwise, but not uniformly, on $[0,1]$. 
Instructions: Do any 4 of the 5 problems. Hand in only the 4 problems you wish to have graded.

1. Let \( K \) be a compact subset of \( \mathbb{R} \), \( x \) a real number not in \( K \). The distance from \( x \) to \( K \) is defined to be

\[
d(x,K) = \inf\{|x - y| : y \in K\}.
\]

Prove that \( d(x,K) > 0 \).

2. Suppose the function \( f : \mathbb{R} \to \mathbb{R} \) satisfies

\[
|f(x) - f(y)| \leq |x - y|^p
\]

for all \( x, y \in \mathbb{R} \), where \( p > 1 \) is a constant. Prove that \( f \) is constant.

3. a) State the definition of Riemann integrability for an arbitrary function \( f : [a,b] \to \mathbb{R} \).

b) Prove, using only the definition in part (a), that the function

\[
f(x) = \begin{cases} 
0, & x = 0 \\
\frac{1}{x}, & 0 < x \leq 1 
\end{cases}
\]

is not Riemann integrable on \([0,1]\).

4. Let \( \sum_{n=1}^{\infty} a_n \) be a convergent series of non-negative constants, and let

\[
s_n = a_1 + a_2 + \ldots + a_n, \quad n \geq 1.
\]

Under what conditions does \( \sum_{n=1}^{\infty} s_n \) converge? Give a rigorous justification for your answer.

5. Suppose \( \{f_n\} \) and \( \{g_n\} \) are sequences of continuous functions on \([a,b]\) such that \( f_n \to f \) uniformly on \([a,b]\) and \( g_n \to g \) uniformly on \([a,b]\). Prove that

\[
f_n g_n \to fg
\]

uniformly on \([a,b]\).
Do any 4 of the following problems.

Hand in only the 4 problems you wish to have graded.

1. Prove or disprove: If \( A \subseteq \mathbb{R} \), and \( L \) is the set of all limit points of \( A \), then every limit point of \( L \) must also be a limit point of \( A \).

2. Prove or disprove: If \( f: [0,1] \rightarrow \mathbb{R} \) is continuous, and \( f(x) \) is rational for all \( x \in [0,1] \), then \( f \) is constant in \([0,1]\).

3. Suppose that \( a < b \) and \( f: [a,b] \rightarrow \mathbb{R} \) is continuous. If \( \int_{r}^{s} f(x) \, dx = 0 \) for all \( r,s \in [a,b] \), prove that \( f(x) \equiv 0 \) on \([a,b]\).

4. Suppose that \( \{x_k\} \) is a convergent sequence in \( \mathbb{R} \), with \( \{x_k\} \rightarrow L \). If \( y_k = \frac{x_1 + x_2 + \ldots + x_k}{k} \), prove that the sequence \( \{y_k\} \) also converges to \( L \).

Hint: \[ L - \sum_{j=1}^{k} \frac{x_j}{k} = -\sum_{j=1}^{k} \frac{L - x_j}{k} \]

5. Let \( A \subseteq \mathbb{R} \) and \( f_n: A \rightarrow \mathbb{R} \) for each \( n = 1, 2, 3, \ldots \). Prove that if the series \( \sum_{n=1}^{\infty} f_n(x) \) is uniformly convergent on \( A \), then the sequence \( \{f_n\} \) converges to 0 uniformly on \( A \).
Do any 4 of the 5 problems.
Hand in only the 4 problems you wish to have graded.

I. Let $E$ be an arbitrary non-empty bounded subset of $\mathbb{R}$, and let $b = \sup E$ (= l.u.b. $E$).

Prove that there is a sequence $(x_n)$ of points in $E$ such that $\lim_{n \to \infty} x_n = b$.

II. Suppose that $f : (0, \infty) \to \mathbb{R}$ is uniformly continuous, and that

$$\lim_{n \to \infty} f\left(\frac{1}{n}\right) = L \text{ for some } L \in \mathbb{R}.$$ 

Prove that $\lim_{x \to 0^+} f(x) = L$.

III. Let $a < c_1 < c_2 < \cdots < c_k < b$ and define $f : [a, b] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \neq c_i \text{ for } i = 1, 2, \ldots, k \\ 1 & \text{if } x = c_i \text{ for some } c_i \end{cases}$$

Using only the definition of the integral, prove that $f$ is integrable on $[a, b]$, and find the value of $\int_a^b f(x) \, dx$.

IV. Suppose that $(a_n)$ is a sequence of real numbers such that the series

$$\sum_{n=1}^{\infty} |a_{n+1} - a_n|$$

coversges. Prove that the sequence $(a_n)$ also converges.

V. Decide whether or not the sequence of functions $f_n(x) = \frac{x}{1 + nx^2}$, $n = 1, 2, 3, \ldots$ is uniformly continuous on $\mathbb{R}$, and prove your answer.
M.S. Analysis Exam  
Fall 1991

Do any 4 of the 5 problems.  
Hand in only the 4 problems you wish to have graded.

1. Either of the following two conditions can be used to define what is meant by a closed set $A \subseteq \mathbb{R}$.

   (a) If $\{a_n\}$ is a sequence of points in $A$ that converges to some limit $L \in \mathbb{R}$,  
   then $L \in A$.

   (b) The complement of $A$ is open.

   Prove that condition (a) implies condition (b).

2. Suppose $f : (0,1) \rightarrow \mathbb{R}$ is uniformly continuous.

   Prove that $\lim_{n \rightarrow \infty} f\left(\frac{1}{n}\right)$ exists.

3. If $a < b$, $f : [a,b] \rightarrow [0,\infty)$, and the Riemann integral $\int_{a}^{b} f(x) \, dx$ exists,  
   prove that $f$ is bounded above.

4. Let $a_n = \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n}$ for $n = 1, 2, 3, \cdots$.

   Prove that the sequence $\{a_n\}$ converges.

5. Let $A \subseteq \mathbb{R}$ and let $\{f_n\}$ be a sequence of real-valued functions on $A$  
   converging uniformly to a bounded real-valued function on $A$. Show that there  
   are constants $K$ and $N$ such that $|f_n(x)| \leq K$ for all $x \in A$ whenever $n > N$.  

Do any 4 of the 5 problems.
Hand in only the 4 problems you wish to have graded.

1. Prove or disprove: If $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing and onto, then $f$ is continuous on $\mathbb{R}$.

2. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous, and $K$ is a compact subset of $\mathbb{R}$. Show that $f(K)$ is compact.

3. Suppose that $a < b$ and $f : [a,b] \to \mathbb{R}$ is continuous, and for all $x \in [a,b]$,
\[
\int_{a}^{x} f(t) \, dt = \int_{x}^{b} f(t) \, dt.
\]
Show that $f(x) = 0$ for all $x \in [a,b]$.

4. Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable, and $\lim_{x \to \infty} f'(x) = 0$.
Show that $\lim_{x \to \infty} [f(x + 1) - f(x)] = 0$.

5. Suppose that for each $n = 1, 2, 3, \ldots$, $f_n : \mathbb{R} \to \mathbb{R}$ is continuous and that $f_n \to f$ uniformly on $\mathbb{R}$. Show that $f$ is continuous on $\mathbb{R}$. 
Analysis Exam: Fall 1990

Do any 4 of the 5 problems.
Hand in only the 4 problems you wish to have graded.

1. Suppose \( \{a_n\} \) is a sequence of real numbers and \( \sum_{n=1}^{\infty} a_n \) converges absolutely

   a) Show \( \sum_{n=1}^{\infty} a_{2n} \) converges.

   b) Give an example showing that "absolutely" is necessary in the hypothesis above.

2. Suppose \( a < b \) and \( f \) is Riemann integrable and non-negative on \([a,b]\).
   If there is a \( p \in (a,b) \) such that \( f \) is continuous at \( p \) and \( f(p) > 0 \), show
   \[
   \int_{a}^{b} f(x)dx > 0.
   \]

3. Show that \( f: \mathbb{R} \to \mathbb{R} \) is continuous at \( x = a \) if the sequence \( \{f(x_n)\} \) converges to \( f(a) \)
   whenever the sequence \( \{x_n\} \) converges to \( a \).

4. Let \( A \) be a non-empty closed subset of \( \mathbb{R} \), and \( p \in \mathbb{R} \). Let \( d = \inf_{x \in A} |p - x| \). Prove that
   there is at least one \( y \in A \) with \( |p - y| = d \).

5. a) For each \( n \geq 1 \), show that \( f_n(x) = \frac{1}{1 + nx} \) is uniformly continuous on \([0,\infty)\).

   b) Does the sequence \( \{f_n\} \) \( n \geq 1 \) converge uniformly on \([0,\infty)\)? Justify your answer.
Analysis Exam

Spring 1990

Instructions: Do any 4 of the 5 problems. Hand in only the 4 problems that you wish to be graded. All functions are real valued functions of a single real variable. The problems are intended to be reasonable but not trivial. You are expected to justify your assertions.

1. In Parts (i) through (iii), prove the given statement.
   i) A convergent sequence is bounded.
   ii) A convergent sequence is a Cauchy sequence.
   iii) A monotonically increasing sequence which is bounded above is convergent.

2. Suppose that \( f \) is a continuous function on \([0, \infty)\), and \( \lim_{x \to \infty} f(x) \) exists and is finite.
   Show: \( f \) is uniformly continuous on \([0, \infty)\).

3. Consider the series \( \sum_{n=1}^{\infty} a_n \). Suppose \( a_n \geq 0 \) and that there exists \( k > 1 \) such that \( \lim_{n \to \infty} n^k a_n \) exists and is finite.
   \[ \sum_{n=1}^{\infty} a_n \text{ BDD.} \implies a_n \leq \frac{1}{n^k} \text{ comparison w/ } p\text{-series} \]
   Show: \( \sum_{n=1}^{\infty} a_n \) converges.

4. Suppose \( f \) is Riemann integrable on \([a,b]\).
   Show: For any \( c, d \) such that \( a \leq c < d \leq b \), \( f \) is Riemann integrable on \([c,d]\).

5. Let \( h_n(x) = nx e^{-nx^2} \) for \( x \in [0,1] \).
   i) Find \( \lim_{n \to \infty} h_n(x) \).
   ii) Find \( \lim_{n \to \infty} \int_0^1 h_n(x) \, dx \).
Analysis Exam

Spring 1989

Instructions: Do any four of the five problems. All functions are real valued functions of a single real variable. The problems are intended to be reasonable but not trivial. You are expected to justify your assertions.

1. Find the greatest lower bound of the set

\[ A = \left\{ \left( 1 + \frac{1}{n} \right)^n \mid n = 1, 2, 3, \ldots \right\}. \]

2. Let \( f \) be a real valued function defined and continuous on \( \mathbb{R} \). Suppose that \( f' \), the derivative, exists for all \( x \neq 0 \) and that \( \lim_{x \to 0} f'(x) = 1. \)

Show that \( f'' \) exists at \( x = 0 \) and that \( f''(0) = 1. \)

3. Let \( f(x) = \sum_{n=1}^{\infty} \frac{n \cdot e^{nx}}{3^n} \). For which values of \( x \) is \( f \) continuous?

4. Determine whether \( \sum_{n=1}^{\infty} a_n \) converges or diverges for each of the following:

(i) \( a_n = \frac{n!}{n^n} \)

(ii) \( a_n = \cos \left( \frac{1}{n^2} \right) \)

(iii) \( a_n = (-1)^n \frac{\ln n}{n} \)

(iv) \( a_n = \sin \frac{1}{n} \)

5. Give an example of a real valued function \( f \) on \([0,1]\) that is bounded but not Riemann integrable, and prove your assertion.
Analysis Exam

Fall 1989

Instructions: Do any 4 of the 5 problems. Hand in only the 4 problems that you wish to be graded. All functions are real valued functions of a single real variable. The problems are intended to be reasonable but not trivial. You are expected to justify your assertions.

1. Let $A, B$ be non-empty, bounded sets of reals and let $C = \{(a + b) | a \in A, b \in B\}$.
   **SHOW:** $\sup C = \sup A + \sup B$.

2. Suppose \{a_n\} is a sequence with the following property:
   there exists a fixed real number $q \in (0,1)$ such that $|a_{n+1}| \leq qa_n$ for $n = 1, 2, 3, \cdots$.
   **SHOW:** \{a_n\} converges.

3. In each of Parts (i) through (iii), determine whether the given sequence or series of functions converges uniformly on the indicated interval.

   i) $f_n(x) = \frac{nx}{1 + n^2x^2}; [0,1]$  
   ii) $\sum_{n=1}^{\infty} (x/n) \ln x^n; (0,1]$  
   iii) $\sum_{n=1}^{\infty} x^n; [0,1]$  

4. Let $f: \mathbb{R} \to \mathbb{R}$ be a function satisfying the condition: $f(x + y) = f(x) + f(y)$ for all $x$ and $y$.
   a) Find $f(0)$.
   b) Show that if $\lim_{x \to 0} \frac{f(x)}{x}$ exists and is finite then $f$ is differentiable everywhere.

5. Let $f: [a,b] \to \mathbb{R}$ be an increasing function.
   **SHOW:** $f$ is Riemann-integrable on $[a,b]$. 

Instructions: Do any 4 of the 5 problems. All functions are real valued functions of a single real variable. The problems are intended to be reasonable but not trivial. You are expected to justify your assertions.

1. Let \( f \) be monotonically increasing on the closed interval \([a,b]\). Show that
\[
\lim_{x \to x_0^+} f(x)
\]
exists for each \( x_0 \in (a,b) \).
(Note that the limit is taken "from the left."

2. Which of the following functions are uniformly continuous on the given domain?
   a) \( f(x) = x^2 \) for \( x \in \mathbb{R} \)
   b) \( g(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases} \) for \( x \in \mathbb{R} \)
   c) \( f(x) = x^2 \) for \( x \in [0,1] \)
   d) \( g(x) = e^x \) for \( x \geq 0 \)

3. Prove or disprove:
   If \( f(x) = \begin{cases} \frac{1}{n} & \text{for} \quad \frac{1}{n+1} < x \leq \frac{1}{n}; \quad n = 1,2,3,\ldots \\ 0 & \text{for} \quad x = 0 \end{cases} \)
   
   then \( f \) is Riemann integrable on \([0,1]\).

4. Show that if \( a_n \geq 0 \) for \( n = 1,2,3,\ldots \) and \( \sum a_n \) converges, then \( \sum a_n^2 \) converges. Show by example that \( a_n \geq 0 \) is necessary.

5. Show that
\[
\int_0^\pi \left[ \sum_{n=1}^{\infty} \frac{n \sin(nx)}{e^n} \right] dx = \frac{2e}{e^2 - 1}
\]